Broadly tunable and robust UV laser generation on a chip: supplemental document

1. ANGLE-DEPENDENT POWER DISTRIBUTION OF ČERENKOV SECOND-HARMONIC GENERATION SIGNAL

Based on the definitions in Fig. 1 of the main text, an expression for the signal power generated through Čerenkov second-harmonic generation (CSHG) is derived in the following. The pump mode is assumed to be guided in a waveguide structure along the $\hat{z}$-direction. The corresponding wavevector can be written as:

$$\vec{k}_\mathrm{p} = k_\mathrm{p}\hat{z}. \tag{S1}$$

Since conversion is realized as a radiation mode in the cladding, the signal is not confined to the waveguide. Hence, the directionality of the signal wavevector is parameterized, with respect to the pump-carrying waveguide, in spherical coordinates by an elevation angle θ and an azimuthal angle ϕ:

$$\vec{k}_\mathrm{s} = k_\mathrm{s}\begin{pmatrix}\sin(\theta)\cos(\phi)\\ \sin(\theta)\sin(\phi)\\ \cos(\theta)\end{pmatrix}. \tag{S2}$$

Note that both k_p and k_s are complex quantities containing the wavenumbers and attenuation coefficients for the pump and signal respectively:

$$k_\mathrm{s} = \beta_\mathrm{s} + i\alpha_\mathrm{s}/2, \tag{S3a}$$
$$k_\mathrm{p} = \beta_\mathrm{p} + i\alpha_\mathrm{p}/2. \tag{S3b}$$

With respect to the geometric axes in Fig. 1, a position vector can be defined:

$$\vec{r} = \begin{pmatrix}x\\ y\\ z\end{pmatrix}. \tag{S4}$$

The total electric fields of the pump (guided by the waveguide) and the signal (radiating from the cladding) are defined as:

$$\vec{\mathcal{E}}_\mathrm{p}(\vec{r},t) = e^{i\vec{k}_\mathrm{p}\cdot\vec{r}}e^{-i\omega_\mathrm{p}t}\mathcal{A}_\mathrm{p}(z,t)\vec{\mathfrak{e}}_\mathrm{p}(x,y) \tag{S5a}$$
$$\vec{\mathcal{E}}_\mathrm{s}(\vec{r},t) = e^{-i\omega_\mathrm{s}t}\int_\theta\int_\phi e^{i\vec{k}_\mathrm{s}\cdot\vec{r}}\mathcal{A}_\mathrm{s}(z,t;\theta,\phi)\vec{\mathfrak{e}}_\mathrm{s}(x,y;\theta,\phi)\,\mathrm{d}\theta\,\mathrm{d}\phi. \tag{S5b}$$

The corresponding magnetic field of the signal can be written analogously as:

$$\vec{\mathcal{H}}_\mathrm{s}(\vec{r},t) = e^{-i\omega_\mathrm{s}t}\int_\theta\int_\phi e^{i\vec{k}_\mathrm{s}\cdot\vec{r}}\mathcal{A}_\mathrm{s}(z,t;\theta,\phi)\vec{\mathfrak{h}}_\mathrm{s}(x,y;\theta,\phi)\,\mathrm{d}\theta\,\mathrm{d}\phi. \tag{S5c}$$

In these expressions, ω_p and ω_s are the angular frequencies of the pump and signal respectively, $\vec{\mathfrak{e}}_\mathrm{p}$, $\vec{\mathfrak{e}}_\mathrm{s}$ and $\vec{\mathfrak{h}}_\mathrm{s}$ are transverse electric and magnetic mode profiles, and $\mathcal{A}_\mathrm{p}$ and $\mathcal{A}_\mathrm{s}$ are corresponding amplitudes. Since the radiation modes form a continuum [1], the expression for the total signal field must contain contributions from all possible radiation modes. This is ensured through the parametrization in the elevation and azimuthal angles θ and ϕ. By integrating over all θ's and ϕ's in Eqs. (S5b) and (S5c), contributions from the entire continuum of radiation modes is considered. In the following, cw-operation is assumed, and hence the time-dependence of the amplitude functions is dropped.

The second-harmonic signal is generated through a nonlinear polarization $\vec{\mathcal{P}}_\mathrm{s}$ induced by the pump as it propagates along the waveguide in the $\hat{z}$-direction. The corresponding evolution of the signal amplitude can then be described by a chief equation [2], where the oscillating terms are contained in the amplitude:

$$\partial_z \mathcal{A}_\mathrm{s} = i\frac{\omega_\mathrm{s}}{4N_\mathrm{s}} \iint_{\mathbb{R}^2} \vec{\mathcal{P}}_\mathrm{s} \cdot \vec{\mathfrak{e}}_\mathrm{s}^* \, e^{-i\varphi_\mathrm{s}} \, \mathrm{d}x \, \mathrm{d}y. \tag{S6}$$

For conciseness of notation, a propagator is defined as:

$$\varphi_\mathrm{s} \equiv \vec{k}_\mathrm{s} \cdot \vec{r} - \omega_\mathrm{s} t. \tag{S7}$$

The parameter N_s is a normalization factor with units of power, defined for the continuum of radiation modes as [1]:

$$N_\mathrm{s} = \frac{1}{2} \iint_{\mathbb{R}^2} \Re\left(\left(\vec{\mathfrak{e}}_s(\theta,\phi) \times \vec{\mathfrak{h}}_s^*(\theta',\phi') \right) \cdot \hat{z} \right) \, \mathrm{d}x\mathrm{d}y = P\delta(\theta'-\theta)\,\delta(\phi'-\phi). \tag{S8}$$

The Dirac delta distribution in Eq. (S8) ensures the necessary orthonormality of the radiation modes. The quantity P has units of power, and is by convention set to 1 mW. For a second-order nonlinearity, the induced nonlinear polarization can be expressed as [3]:

$$\vec{\mathcal{P}}_s = \varepsilon_0 D \mathcal{A}_\mathrm{p}^2(z) \vec{v}_\mathrm{s} e^{-i\Delta\varphi_\perp} e^{-i\Delta\varphi_\parallel} e^{i\varphi_s}. \tag{S9}$$

In Eq. (S9), the parameter D is the $\chi^{(2)}$ nonlinear tensor of the materials. For SiO_2 and Si_3N_4 it is vanishing, while it takes the following shape for BBO:

$$D \approx d_{16} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{S10}$$

The approximate equality in Eq. (S10) is due to neglecting the d_{15} and d_{33} coefficients, which in BBO are small compared to the d_{16} coefficient [4]. Additionally, in Eq. (S9), two terms containing the phase mismatch in the longitudinal and transverse directions respectively are introduced:

$$\Delta\varphi_\parallel = \left[k_\mathrm{s}\cos(\theta) - 2k_\mathrm{p}\right] z. \tag{S11a}$$

$$\Delta\varphi_\perp = k_\mathrm{s}\sin(\theta)\left[\cos(\phi)x + \sin(\phi)y\right]. \tag{S11b}$$

By noting that the longitudinal part of the propagator cancels when:

$$\theta = \arccos\left(2k_\mathrm{p}/k_\mathrm{s}\right), \tag{S12}$$

the commonly stated criterion for CSHG is obtained. This criterion relates the effective index of the pump n_p and refractive index of the signal n_s to the Cerenkov angle θ_C:

$$\theta_\mathrm{C} = \arccos\left(n_\mathrm{p}/n_\mathrm{s}\right). \tag{S13}$$

Eq. (S13) is of great importance when designing devices based on CSHG emission. Returning to Eq. (S9), The vector $\vec{v}_\mathrm{s}$ contains the pump field mode profiles, defined based on the mode field components along the crystal axes of the BBO cladding as:

$$\vec{v}_\mathrm{s} = \begin{pmatrix} \mathfrak{e}_{\mathrm{p},X}^2 \\ \mathfrak{e}_{\mathrm{p},Y}^2 \\ \mathfrak{e}_{\mathrm{p},Z}^2 \\ 2\mathfrak{e}_{\mathrm{p},Y}\mathfrak{e}_{\mathrm{p},Z} \\ 2\mathfrak{e}_{\mathrm{p},X}\mathfrak{e}_{\mathrm{p},Z} \\ 2\mathfrak{e}_{\mathrm{p},X}\mathfrak{e}_{\mathrm{p},Y} \end{pmatrix}. \tag{S14}$$

Inserting Eq. (S9) into the chief equation given by Eq. (S6), and using the normalization in Eq. (S8), it can be shown that the signal amplitude of a radiation mode given by any combination of θ and ϕ can be expressed as:

$$\partial_z \mathcal{A}_{\mathrm{s}}|_{\theta,\phi} = i\frac{\varepsilon_0\omega_{\mathrm{s}}}{4N_{\mathrm{s}}}\mathcal{A}_{\mathrm{p}}^2(z)e^{-i\Delta\varphi_\parallel}\iint_{\mathbb{R}^2} D\vec{v}_{\mathrm{s}}\cdot\vec{\mathfrak{e}}_{\mathrm{s}}^*\left(\theta',\phi'\right)e^{-i\Delta\varphi_\perp}\,\mathrm{d}x\mathrm{d}y. \tag{S15}$$

In the above expression, $\theta \neq \theta'$ and $\phi \neq \phi'$. By integrating Eq. (S15) over all angles θ and ϕ, including θ' and ϕ' and using Eq. (S8), an expression for the signal amplitude at specifically θ' and ϕ' is obtained:

$$\partial_z \mathcal{A}_{\mathrm{s}}|_{\theta',\phi'} = i\frac{\varepsilon_0\omega_{\mathrm{s}}}{4P}\mathcal{A}_{\mathrm{p}}^2(z)e^{-i\Delta\varphi_\parallel}|_{\theta'}\iint_{\mathbb{R}^2} D\vec{v}_{\mathrm{s}}\cdot\vec{\mathfrak{e}}_{\mathrm{s}}^*(\theta',\phi')\,\mathcal{R}(x,y)\;\mathrm{d}x\,\mathrm{d}y. \tag{S16}$$

In Eq. (S16), the term $\mathcal{R}(x,y)$ handles the transverse part of the phase mismatch and is defined as:

$$\mathcal{R}(x,y) = \int_\theta\int_\phi e^{-i\Delta\varphi_\perp}|_{\theta,\phi}\,\mathrm{d}\theta\,\mathrm{d}\phi. \tag{S17}$$

From Eq. (S16), a z-independent and unit-less coupling coefficient can be defined, indicating the strength of coupling from the guided pump mode into a second-harmonic radiation mode signal given by θ' and ϕ':

$$\kappa_{\mathrm{s}}(\theta',\phi') = \frac{c\varepsilon_0}{4P}\iint_{\mathbb{R}^2} D\vec{v}_{\mathrm{s}}\cdot\vec{\mathfrak{e}}_{\mathrm{s}}^*(\theta',\phi')\mathcal{R}(x,y)\,\mathrm{d}x\,\mathrm{d}y. \tag{S18}$$

Apparent from Eq. (S18), the coupling coefficient is obtained by integrating the interaction of the second-order nonlinear tensor element, pump mode, and signal mode over the entire transverse plane. Since the guided pump mode decays evanescently away from the waveguide, this is approximated by performing the calculation numerically over a waveguide cross section encompassing the entirety of the pump mode. By substituting this expression into Eq. (S16), the signal amplitude for a radiation mode given by any θ,ϕ-pair becomes:

$$\partial_z \mathcal{A}_{\mathrm{s}}|_{\theta,\phi} = i\frac{\omega_{\mathrm{s}}}{c}\mathcal{A}_{\mathrm{p}}^2(z)\,e^{-i\Delta\varphi_\parallel}|_\theta\,\kappa(\theta,\phi). \tag{S19}$$

The propagation loss of the pump due to scattering and absorption is contained in the parallel part of the propagator through Eq. (S3a):

$$e^{-i\Delta\varphi_\parallel} = e^{-\alpha_p z}e^{-i(\beta_s\cos\theta - 2\beta_p)z}. \tag{S20}$$

In Eq. (S20), loss of the signal is neglected, i.e. $\alpha_s = 0$. In the following, the pump loss in Eq. (S20) is contained in the z-dependent pump amplitude $\mathcal{A}_{\mathrm{p}}(z)$. Coupling from the signal mode back to the pump is not accounted for, since the signal travels away from the waveguide as it is generated, preventing further interaction. The signal amplitude of a radiation mode parameterized by θ and ϕ after propagating a distance L along the waveguide is then given by:

$$\mathcal{A}_{\mathrm{s}}|_{\theta,\phi}(L) = i\frac{\omega_{\mathrm{s}}}{c}\kappa(\theta,\phi)\int_0^L \mathcal{A}_{\mathrm{p}}^2(z)\,e^{-i(\beta_s\cos\theta-2\beta_p)z}\,\mathrm{d}z. \tag{S21}$$

It is important to note that due to the continuum-nature of the radiation modes, all possible radiation modes are excited simultaneously. Hence, to evaluate the total signal generated, coupling into all possible θ,ϕ-pairs must be considered. This is achieved by calculating the coupling coefficient given by Eq. (S18) in the angular ranges $0 \leq \theta \leq \pi/2$ and $0 \leq \phi \leq \pi/2$, and subsequently the corresponding amplitudes given by Eq. (S21). From these amplitudes, the power of the generated radiation mode signal for a given pump propagation length L can be calculated. The spatially-dependent power flow in the $\hat{z}$-direction along the pump is given by the time-averaged Poynting vector, which by integration over the transverse plane yields the total generated power:

$$P_{\mathrm{s}} = \frac{1}{2}\iint_{\mathbb{R}^2}\Re\left[\vec{\mathcal{E}}_s\times\vec{\mathcal{H}}_s^*\right]\cdot\hat{z}\,\mathrm{d}x\mathrm{d}y. \tag{S22}$$

Inserting the expressions for the total fields given by Eq. (S5b) and Eq. (S5c) into Eq. (S22) yields a z-dependent quantity:

$$\begin{aligned} P_{\mathrm{s}}(z) =& \frac{1}{2}\Re[\int_\theta\int_\phi\int_{\theta'}\int_{\phi'}\mathcal{A}_{\mathrm{s}}(z;\theta,\phi)\mathcal{A}_{\mathrm{s}}^*(z;\theta',\phi')\,\cdot \\ & \iint_{\mathbb{R}^2}\vec{\mathfrak{e}}_{\mathrm{s}}(x,y;\theta,\phi)\times\vec{\mathfrak{h}}_{\mathrm{s}}^*(x,y;\theta',\phi')\cdot\hat{z}\,\mathrm{d}x\mathrm{d}y\mathrm{d}\theta\mathrm{d}\phi\mathrm{d}\theta'\mathrm{d}\phi']. \end{aligned} \tag{S23}$$

Substitution of Eq. (S8) into Eq. (S23), and setting $z = L$, results in the expression:

$$P_{\mathrm{s}}(L) = \Re\left[\int_{\theta}\int_{\phi}\int_{\theta'}\int_{\phi'} \mathcal{A}_{\mathrm{s}}(L;\theta,\phi)\mathcal{A}_{\mathrm{s}}^{*}(L;\theta',\phi')N_{\mathrm{s}}\mathrm{d}\theta\mathrm{d}\phi\mathrm{d}\theta'\mathrm{d}\phi'\right]. \tag{S24}$$

Inserting the RHS of Eq. (S8) yields delta functions in both the elevation and azimuthal angles:

$$P_{\mathrm{s}}(L) = \Re\left[P\int_{\theta}\int_{\phi}\int_{\theta'}\int_{\phi'} \mathcal{A}_{\mathrm{s}}(L;\theta,\phi)\mathcal{A}_{\mathrm{s}}^{*}(L;\theta',\phi')\delta(\theta'-\theta)\delta(\phi'-\phi)\mathrm{d}\theta\mathrm{d}\phi\mathrm{d}\theta'\mathrm{d}\phi'\right]. \tag{S25}$$

By evaluating the first two integrals over θ' and ϕ', the delta functions are eliminated and a final expression for the total radiated power is obtained:

$$P_{\mathrm{s}}(L) = \Re\left[P\int_{\theta}\int_{\phi} \mathcal{A}_{\mathrm{s}}(L;\theta,\phi)\mathcal{A}_{\mathrm{s}}^{*}(L;\theta,\phi)\mathrm{d}\theta\mathrm{d}\phi\right]. \tag{S26}$$

Eq. (S26) describes the total power generated through CSHG by coupling from the guided pump mode to the signal radiation modes. The expression considers the contribution from all radiation modes which constitute the total signal field, and yields a single number expressing the generated power. Using Eq. (S21) for numerical calculation of the signal amplitudes, Eq. (S26) can readily be calculated for a given device geometry.

To retain spatial information on the signal distribution in the angular domains, integration over θ and ϕ can be dropped in Eq. (S26). This yields an expression for the power in each radiation mode defined by θ and ϕ:

$$P_{\mathrm{s}}(L;\theta,\phi) = \Re\left[P\mathcal{A}_{\mathrm{s}}(L;\theta,\phi)\mathcal{A}_{\mathrm{s}}^{*}(L;\theta,\phi)\right]. \tag{S27}$$

Eq. (S27) is used in the simulations in the main text to predict the angular distribution of the generated CSHG signals.

2. DISPERSION MODEL

Through ellipsometry, a Cauchy model of the refractive index of a reference Si_3N_4 wafer is measured. The reference wafer thin-film is deposited directly on silicon in the same Si_3N_4 deposition run as the device wafers used in fabrication. The measured Cauchy model is described by Eq. (S28) with parameters given in table S1. In figure S1, the measured model is compared to reference [5], used in the design simulations.

$$n(\lambda) = A + 10^{2}\frac{B}{\lambda^{2}} + 10^{7}\frac{C}{\lambda^{4}} \tag{S28}$$

A	B	C
$1.976 \pm 0.14 \cdot 10^{-4}$	83.2 ± 0.0673	70.3 ± 0.0369

Table S1. Measured parameters for Eq. (S28) .

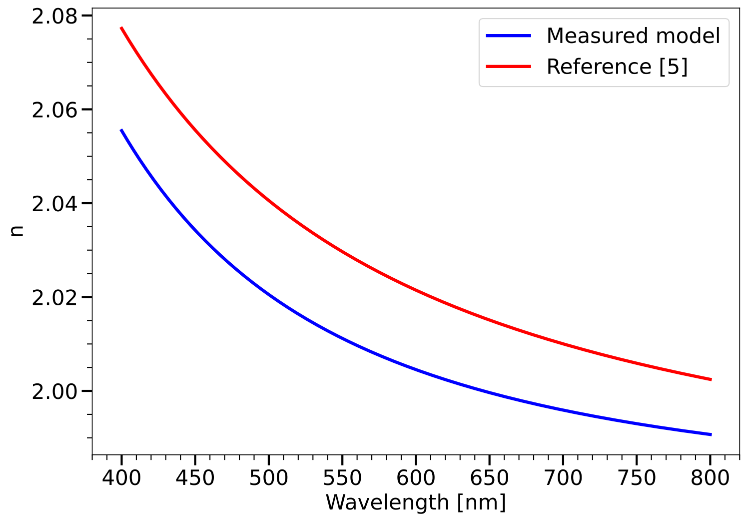

Fig. S1. Comparison of the Si_3N_4 model measured in this work and recent literature [5].

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