Confocal ellipsoidal displaced Gregorian structure for stand-off millimeter-wave imaging: supplemental document

To gain a better understanding of how to align different coordinates of the proposed geometry, a more detailed trigonometric analysis is presented. These mathematical observations assist the readers in constructing the proposed configuration and obtaining more precise ray tracing results.

1. REVISITING DEFINITIONS FOR THE MODIFIED SM CURVE

A more general geometry for the SM curve of Fig. 2 with some additional details is depicted in Fig. S1 where some of the basic geometrical definitions of the constructive curves of the SM are marked on the complete curves of the two displaced and tilted ellipses.

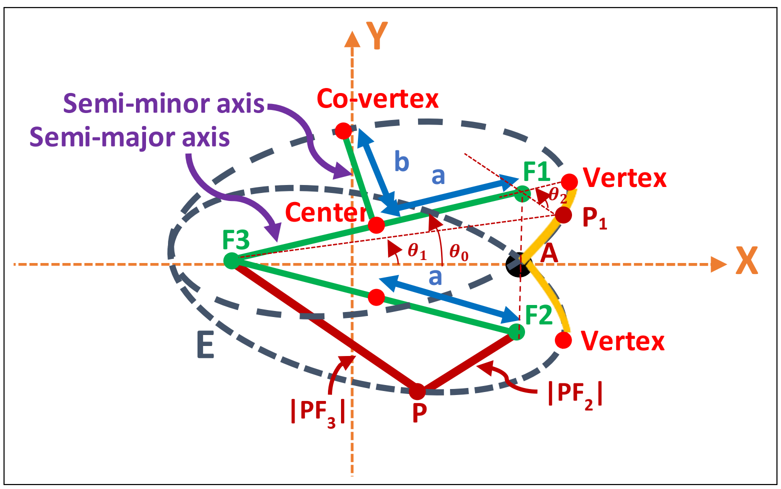

Fig. S1. General geometrical definitions of the two displaced and tilted ellipses constructing the SM curve.

For the sake of simplicity, only some of the geometrical definitions and notations are shown in Fig. S1. First, the lower ellipse is selected as an example. Similar definitions can also be applied to the upper ellipse. Based on the basic trigonometry principles [1], the lower ellipse can be defined as the locus of points in the Euclidean plane regarding the fixed points F_2 and F_3 where:

$$\mathbf{E} = \{\mathbf{P} \,\epsilon\, \Re^2 |\, |\overline{PF_2}| + |\overline{PF_3}| = 2a\} \tag{S1}$$

This statement claims that the ellipse **E** is the set of points P in the $\Re^2$ space such that the sum of the distances $|PF_2|$ and $|PF_3|$ is equal to 2a, where a is defined as the semi-major axis of the ellipse. This argument also dictates that F_2 and F_3 are two foci of the ellipse **E** and the diverging bundle of rays originating from one of them, after reflection, converges at the other one. Other definitions of a standard ellipse, such as linear eccentricity, semi-latus rectum, etc., are also applicable in the geometry of Fig. S1.

If in step (2) of Fig. 2 of the paper, the x-coordinates of the points V_1 and V_2 are selected equal to the x-coordinates of the first foci or $a - c$, then the distance from the apex of the SM to its first foci which is coincidental with the focus of the MM is:

$$\overline{AF_1} = \overline{AF_2} = \frac{b^2}{a} \tag{S2}$$

where this distance, by definition, is also equal to the semi-latus rectum length or the radius of curvature at the vertices in a standard ellipse. By using the Pythagorean theorem, the distance from the apex of the SM to its second foci which is coincidental with the phase center of the source or detector is:

$$|\overline{F_1F_3}| = |\overline{F_2F_3}| = \sqrt{4c^2 + \left(\frac{b^2}{a}\right)^2} \tag{S3}$$

Other geometrical coordinates can also be calculated similarly by using simple trigonometric laws and identities. Therefore, the coordinates of the proposed SM geometry can be easily extracted by only identifying the basic parameters of the standard ellipse.

2. THE SM CURVE IN CARTESIAN AND POLAR COORDINATES

From an analytical geometry point of view, the constructed SM curve can be described in both the Cartesian and polar coordinate systems. For the Cartesian description, the standard ellipse of the step (1) of Fig. 2 of the paper can be described in the xy-plane in its general form as:

$$\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a \tag{S4}$$

which is the canonical equation of a standard ellipse in the Euclidean space. By applying an affine transformation of the coordinates (x,y) as [2]:

$$\begin{cases} x = (X - x_0)\cos\theta + (Y - y_0)\sin\theta \\ y = -(X - x_0)\sin\theta + (Y - y_0)\cos\theta \end{cases} \tag{S5}$$

By simplifying these equations using algebraic identities and categorizing the results using matrix notation, we obtain:

$$\mathbf{E}(X, Y) = \mathbf{V}^T M_Q \mathbf{V} \tag{S6}$$

where $\mathbf{V}$ is the projective coordinates which is defined as:

$$\mathbf{V} = \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} \tag{S7}$$

and the M_Q matrix is calculated as:

$$M_Q = \begin{bmatrix} A & B/2 & D/2 \\ B/2 & C & E/2 \\ D/2 & E/2 & F \end{bmatrix} \tag{S8}$$

The coefficients A-F for the proposed geometry of the paper can be calculated as:

$$\begin{cases} A = (a\,\sin\theta_{CEDG})^2 + (b\,\cos\theta_{CEDG})^2 \\ B = 2(b^2 - a^2)\sin\theta_{CEDG}\cos\theta_{CEDG} \\ C = (a\,\cos\theta_{CEDG})^2 + (b\,\sin\theta_{CEDG})^2 \\ D = -2Ax_c - By_c \\ E = -Bx_c - 2Cy_c \\ F = Ax_c^2 + BCx_cy_c + Cy_c^2 - (ab)^2 \end{cases} \tag{S9}$$

where (x_c, y_c) is the center coordinates of the ellipse. It should be noted that Eq. S6 is the abstract and implicit definition form of a generalized non-degenerate ellipse as a quadratic polynomial set of points (X, Y) on the Cartesian plane provided that:

$$B^2 - 4AC < 0 \tag{S10}$$

In its general form, the curve $E(X, Y)$ is a non-degenerate conic section if:

$$det[M_Q] \neq 0 \tag{S11}$$

where the determinant of matrix M_Q is invariant with respect to both translation and rotation. Furthermore, $E(X, Y)$ is an ellipse if and only if:

$$det[A_{22}] > 0 \tag{S12}$$

where A_{22} is a submatrix of order two which is obtained by removing the third row and column of the matrix M_Q. Moreover, $E(X, Y)$ is a real ellipse if:

$$(A + C)det[M_Q] < 0 \tag{S13}$$

but an imaginary ellipse if:

$$(A + C)det[M_Q] > 0 \tag{S14}$$

To find the center coordinates, we refer to the basic definition of a central point in an elliptical curve. First, we point out that there exists a center point for the ellipse if:

$$det[A_{22}] \neq 0 \tag{S15}$$

Next, we notice that, by definition, any chords of the ellipse pass through it bisect at its central point. In order to calculate this point, [3] suggests to calculate:

$$(x_c, y_c) = \nabla \mathbf{E}(X, Y) = \left[\frac{\partial \mathbf{E}(X, Y)}{\partial X}, \frac{\partial \mathbf{E}(X, Y)}{\partial Y} \right] \tag{S16}$$

which states that the gradient of the quadratic form of the ellipse $\mathbf{E}(X, Y)$ vanishes at the center. For the final SM geometry which is illustrated in step (3) of Fig. 2 of the paper, the center point coordinates can be calculated as:

$$\begin{cases} x_c = a - c + d_{fc} \cos^2 \theta_{CEDG} \\ y_c = \frac{b^2}{a} - d_{fc} \sin \theta_{CEDG} \end{cases} \tag{S17}$$

where d_{fc} is the distance between the foci and the center point of the ellipse which is defined as:

$$d_{fc} = \sqrt{c^2 + \left(\frac{b^2}{a} \right)^2} \tag{S18}$$

Furthermore, Eq. S6 in the new center coordinates can be written using matrix notation in the form of:

$$\begin{pmatrix} x - x_c & y - y_c \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x - x_c \\ y - y_c \end{pmatrix} = \kappa \tag{S19}$$

where constant κ is defined as:

$$\kappa = -\frac{det[M_Q]}{det[A_{22}]} \tag{S20}$$

In this new form, a non-degenerate ellipse corresponds to the condition $\kappa \neq 0$ and $AC > (B/2)^2$. The ellipse is real if the sign of κ be equals to the sign of $(A + C)$. Finally, for the polar coordinates, the SM curve of Fig. S1 can be described parametrically as:

$$\begin{cases} \overline{F_3 P_1} = \frac{a(1-e^2)}{1-e\cos(\theta_0 - \theta_1)} \\ \overline{F_1 P_1} = \frac{a(1-e^2)}{1+e\cos(\theta_2)} \end{cases} \tag{S21}$$

where angles θ_0, θ_1 and θ_2 are defined in Fig. S1.

3. CANONICAL EQUATION OF THE SM CURVE IN THE NEW COORDINATES

To obtain a standard canonical equation for the new displaced and tilted ellipse, first, the ellipse translation is quantified by the vector:

$$\vec{t} = \begin{pmatrix} x_c \\ y_c \end{pmatrix} \tag{S22}$$

as is depicted in Fig. S2.

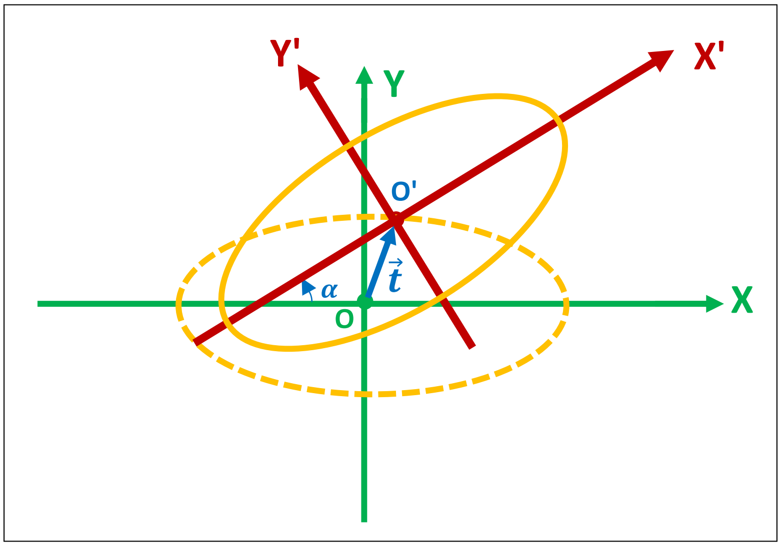

Fig. S2. Defining the displaced and tilted ellipse in the new $x'y'$-coordinates by applying proper translation and rotation on the ellipse.

For applying the rotation by angle α, it is useful to first diagonalize the submatrix A_{22}. The descriptive equation of the new ellipse in the $x'y'$-coordinate system can be written as [3]:

$$\lambda_1 \, x'^2 + \lambda_2 \, y'^2 = -\frac{det[M_Q]}{det[A_{22}]} \tag{S23}$$

where λ_1 and λ_2 are two non-zero eigenvalues of A_{22} submatrix. In the case of a real ellipse, λ_1 and λ_2 have the same algebraic sign [4]. By dividing this equation by κ, defined in Eq. S20, the standard canonical form of an ellipse is obtained as Eq. S4.

REFERENCES

1. T. M. Apostol and M. A. Mnatsakanian, *New horizons in geometry*, vol. 47 (American Mathematical Soc., 2017).
2. J. D. Lawrence, *A catalog of special plane curves* (Courier Corporation, 2013).
3. A. B. Ayoub, "The central conic sections revisited," Math. Mag. **66**, 322–325 (1993).
4. A. J. Pettofrezzo, *Matrices and transformations* (Courier Corporation, 1978).