PROGRAMMABLE KERR COMBS IN LASER CAVITY SOLITON MODE-LOCKED LASER: SUPPLEMENTARY

Section S1- Simulation parameters and normalization

The programmable nested cavities MLL can be simulated based on the following coupled mode equations:

$$\frac{\partial\psi_a}{\partial t} = \left[-\frac{\kappa_a}{\kappa_b} - id_{2a}\frac{\partial^2}{\partial\phi^2} + d_{3a}\frac{\partial^3}{\partial\phi^3} + i\left(|\psi_a|^2 + \frac{2}{\phi_0}\int_{-\phi_0/2}^{\phi_0/2}|\psi_a|^2 d\phi\right)\right]\psi_a + k_{b\to a}\psi_b$$
$$\frac{\partial\psi_b}{\partial t} = \left[-\left(1 + \frac{2}{\kappa_b.T_b} + i\delta_0\right) - d_{1b}\frac{\partial}{\partial\phi} - \frac{i}{2}(2\,d_{2b} - i\sigma_b)\frac{\partial^2}{\partial\phi^2} + d_{3b}\frac{\partial^3}{\partial\phi^3} + \frac{g_{0,EDFA}}{1 + \left|\frac{\psi_b}{\psi_{b,sat}}\right|^2}\right]\psi_b + h_{PDL}(\phi) * \psi_b + k_{a\to b}\psi_a \qquad (S1)$$

The spatial dimension ϕ is normalized relative to the second order chromatic dispersion of the auxiliary cavity. ϕ_0 denotes the normalized spatial period. Thus, the amplitude of the electric field was expressed through the Kerr coefficient of the auxiliary cavity by:

$$\psi_a = \sqrt{\frac{\gamma_a}{\kappa_b} {n_0}^2 {v_g}^2 \varepsilon_0 A_{eff}}.E_a$$
$$\psi_b = \sqrt{\frac{\gamma_a}{\kappa_b} {n_0}^2 {v_g}^2 \varepsilon_0 A_{eff}}.E_b \qquad (S2)$$

E is the electric field ([V/m]), γ_a represents the Kerr coefficient ([/W/m]), ε_0 the vaccum permittivity ([F/m]), n_0 the fiber effective refractive index and A_{eff} the fiber cross section ([/m²]). v_g is the group velocity ([m/s]), κ_b the damping rate of the main cavity ([/s]) related to the loss α_b (Neper) of the main cavity by $\kappa_b = \alpha_b/T_b$ (T_b is the main cavity round trip [s], L_b the related longitudinal dimension of the cavity [m].). It is to notice that as the resonator considered is a fibered SMF28 Fabry-Pérot resonator, no distinction has been made on the effective cross section used to normalize the main field (ψ_b) and the auxiliary one (ψ_a).

The spatial dimension x is related to ϕ by:

$$\phi = \frac{2\pi}{T_a} v_g x \sqrt{\frac{\kappa_b}{2|D_{2,a}|}} \qquad (S3)$$

With T_a the round-trip time of the auxiliary cavity and $D_{2,a}$ the second order chromatic given by:

$$D_{2,a} = -\frac{\beta_{2,a}}{\beta_1}\left(\frac{2\pi}{T_a}\right)^2 \qquad (S4)$$

$\beta_1 = 1/v_g$ ([s/m]) and β_{2a} having the dimension [s²/m]. In general, the chromatic dispersion of nth order ($D_{n,a}$) is given by ($n \geq 2$):

$$D_{n,a} = -\frac{\beta_{n,a}}{\beta_1}\left(\frac{2\pi}{T_a}\right)^n \qquad (S5)$$

and for consistence with the definition of ϕ, rewritten on (S1) by:

$$d_{n,a} = \frac{1}{n!} D_{n,a} \frac{2}{\kappa_b}\left(\frac{\kappa_b}{2D_{2,a}}\right)^{\frac{n}{2}} \qquad (S6)$$

The same normalization has been carried out for the main cavity by simply changing the $\beta_{n,a}$ sequence with the chromatic dispersion coefficients of the main cavity $\beta_{n,b}$, and providing the $d_{n,b}$ present in (S1). The detuning between both cavities Free Spectral Range (FSR) is carried by ζ_b, which is related to the differential delay $\Delta\tau$ by:

$$\zeta_b = \frac{\delta\beta_1}{\beta_1} . \frac{2\pi}{T_a} \qquad (S7)$$

With $\delta\beta_1 = \Delta\tau / L_b$ [s/m] and $\Delta\tau = (FSR_a - FSR_b)/{FSR_a}^2$.

The time dimension (t) provides the scale of the dynamic. We chose to normalize the time dimension relative to the damping ratio of the main cavity (κ_b), as indicated by the loss unity for the main cavity in (S1). The coefficient $2/\kappa_b T_b$ is present to retrieve the field ψ_b due to the transmission through the auxiliary cavity of the electric field circulating in the main cavity, as proposed in [1].
The relative resonant frequencies of the auxiliary and main cavity need also to be considered for the soliton laser to be established. To do this, we consider a mode of reference in the auxiliary cavity $\nu_{0,a}$. The closest resonance of the main cavity, can then be described relatively to $\nu_{0,a}$ by:

$$\upsilon_{0,b} - \upsilon_{0,a} = -FSR_{b.}\Delta \qquad (S8)$$

so that Δ represents the detuning of the main cavity resonances relative to its free spectral range. The carrier phase shift due to this resonance detuning is taken in consideration by the δ_0 coefficient:

$$\delta_0 = 2\pi . \Delta . \frac{2}{\kappa_b T_b} \qquad (S9)$$

The main cavity also includes an Erbium Doped Fiber Amplifier (EDFA), an optical filter and the Programmable Delay-Line (PDL). The gaussian filter being much narrower than the EDFA bandwidth, the EDFA was modelled by a flat gain, homogeneously saturable amplifier medium. The small signal gain was normalized relatively to the damping rate of the main cavity:

$$g_{0,EDFA} = \frac{G_{0,EDFA}}{\kappa_b T_b} \qquad (S10)$$

with $G_{0,EDFA}$ expressed in Neper. Similarly, the gaussian filter coefficient σ_b arises from the spatial and temporal normalization. As a result, we obtain:

$$\sigma_b = \frac{1}{2\Gamma_f^2 T_b}\sqrt{\frac{2}{\kappa_b\left|D_{2,a}\right|}}\frac{2\pi}{T_a} \qquad (S11)$$

With Γ_b the filter e^{-1} bandwidth in [/s]. The last component of the main cavity is the PDL. Only the two first orders of the chromatic dispersion ($\beta_{0,PDL}$, $\beta_{1,PDL}$) were taken into account, mainly due to the short lengths of the PDL delay-line regarding the pulse width obtained at steady-state, making higher orders negligible. Following the same normalization as before, we obtain for the nth delay-line:

$$\begin{aligned} d_{0,PDL}^n &= \omega_0 \tau_{n,PDL}\sqrt{\frac{\kappa_b}{2\left|D_{2,a}\right|}} \\ d_{1,PDL}^n &= \frac{\tau_{n,PDL}}{T_b}\frac{2\pi}{T_a}\sqrt{\frac{\kappa_b}{2\left|D_{2,a}\right|}} \end{aligned} \qquad (S12)$$

The set $\left\{d_{0,PDL}^n, d_{1,PDL}^n\right\}_{1\leq n\leq 8}$ of the 8 delay-lines are then injected into the sequence of transfer functions (2), operation included on (S1) by the impulse response $h_{PDL}(\phi)$.

Finally, the two cavities are coupled together by the coefficient $\{k_{a\to b}, k_{b\to a}\}$, due to the external coupling of the auxiliary cavity. If we define θ as the loss of the high Q auxiliary cavity, induced by the external coupling, then $\theta_{ext} = \pi\Gamma_{aux}T_b$ (Γ_{aux} being the Full Width at Half Maximum of the auxiliary cavity resonances), and:

$$k_{a\to b} = k_{b\to a} = \sqrt{\theta_{ext}}\frac{2}{\kappa_b T_b} \qquad (S13)$$

Table (S1) summarizes all physical parameters of the MLL simulated and relates physical unit to their normalized counterparts. Three parameters were varied to carry simulations: $G_{0,EDFA}$, $P_{b,sat}$ and Δ. Experimentally, both $G_{0,EDFA}$ and $P_{b,sat}$ can be changed independently by adjusting the pumping current of the EDFA and the optical loss of the main cavity. The resonance detuning Δ need to be varied typically with a fiber stretcher in the main cavity, or by a thermal adjustment of the auxiliary cavity.

Table S1. Cavities parameters used for simulation

Main cavity	Value	unit	Norm	value
L_b	20.88	m		
T_b	102.4	ns		
FSR_b	9.765	MHz		
α_b	3	dB	κ_b	7.77 10^6
$\beta_{2,b}$	-5.25	ps^2/km	$d_{2,b}$	0.12
$\beta_{3,b}$	0.152	ps^3/km	$d_{3,b}$	-1.1 10^{-3}
$\delta\beta_1$ (delay detuning)	0	ps/m	ζ_b	0
$(\upsilon_{0,b} - \upsilon_{0,a})/FSR_b$ (resonance detuning)	$0.4 \leq \Delta \leq 0.55$		δ_0	$6.31 \leq \delta_0 \leq 8.68$
$G_{0,EDFA}$	$14 \leq G_{0,EDFA} \leq 20$	dB	$g_{0,EDFA}$	$4 \leq g_{0,EDFA} \leq 5.78$
$P_{b,sat}$	$14 \leq P_{b,sat} \leq 18$	dBm	$\lvert\psi_{b,sat}\rvert^2$	$0.16 \leq \lvert\psi_{b,sat}\rvert^2 \leq 0.4$
$\lambda_{c,filter}$	1560	nm		
$\Delta\lambda_{filter}$	5	nm	σ_b	0.17
Auxiliary cavity	**Value**	**unit**	**Norm**	**value**
L_a	0.2088	m		
T_a	1024	ps		
FSR_a	0.9765	GHz		
$\beta_{2,a}$	-21.96	ps^2/km	$d_{2,a}$	0.5
$\beta_{3,a}$	0.152	ps^3/km	$d_{3,a}$	-1.1 10^{-3}
γ_a	1.23	/W/km		
α_a	0.07	dB	κ_a	3.14 10^7
Γ_{aux} (external coupling)	5	MHz	$k_{a\to b}$ $k_{b\to a}$	3.18
η (coupling ratio)	0.4			

Section S2- Simulation conditions

Simulations were carried on based on an initial white noise (uniform power spectral density with random phase) and simulation performed until a steady-state is reached. To complete the proposed model, supermodes competition could also be considered by replacing the single equation of the main cavity field (ψ_b) by a set of cavity fields ($\psi_{b,q}$) associated with all main cavity's resonances ($\upsilon_{0,b}^q$) included inside the auxiliary cavity main mode FMHW. Thus, equation (S7) becomes:

$$\upsilon_{0,b}^q - \upsilon_{0,a} = -FSR_{b.}\Delta^q < \Gamma_{aux} \quad (S14)$$

In our case, the simulation conditions present a FSR_b twice larger than the FWHM of the auxiliary cavity, so that the supermode competition has been neglected.

Section S3 - Fourier transform for a pure delay interleaving operation

Equation (3) is the Fourier transform for a pure delay interleaving operation, ideally obtained by the PDL without carrier phase step. The transfer function is obtained based on the assumption that the set of voltage applied on the PDL leads to a direct division of the repetition pulse train by 2^l, l being an integer. As described in the article, this condition is not mandatory to obtain a harmonic oscillation of the MLL, but allows a simple sequence between the delays used to multiply the repetition rate. Thus, assuming a pulse interleaving of pure delay (i.e. without loss), with a minimal delay $\tau = T_{rep}/2^l$, the PDL impulse response is given by:

$$h(t) = \sum_{k=0}^{2^l-1} \delta\left(t - k\frac{T_{rep}}{2^l}\right) \tag{S15}$$

leading to the Fourier transform:

$$\begin{aligned}
H(i\omega) &= \sum_{k=0}^{2^l-1} \exp\left(ik\frac{T_{rep}}{2^l}\omega\right) \\
&= exp\left(i\frac{1}{2}\left(1-\frac{1}{2^l}\right)T_{rep}\omega\right)\left[\sum_{k=0}^{2^{l-1}-1} exp\left(i\left(k-\frac{1}{2}\left(1-\frac{1}{2^l}\right)\right)\frac{T_{rep}}{2^l}\omega\right)\right. \\
&\qquad \left. + \sum_{k=2^{l-1}}^{2^l-1} exp\left(i\left(k-\frac{1}{2}\left(1-\frac{1}{2^l}\right)\right)\frac{T_{rep}}{2^l}\omega\right)\right] \\
&= \exp\left(i\frac{1}{2}\left(1-\frac{1}{2^l}\right)T_{rep}\omega\right)\sum_{p=0}^{2^{l-1}-1} exp\left(-i(2p+1)\frac{T_{rep}}{2^{l+1}}\omega\right) \\
&\quad + exp\left(i(2p+1)\frac{T_{rep}}{2^{l+1}}\omega\right)
\end{aligned}$$

$$H(i\omega) = 2\, exp\left(i\frac{1}{2}\left(1-\frac{1}{2^l}\right)T_{rep}\omega\right)\sum_{p=0}^{2^{l-1}-1} cos\left((2p+1)\frac{T_{rep}}{2^{l+1}}\omega\right) \tag{S16}$$

References

1. H. Bao, A. Cooper, M. Rowley, L.D. Lauro, J.S.T. Gongora, S.T. Chu, B.E. Little, G.L. Oppo,R. Morandotti, D.J. Moss, B.Wetzel, M. Peccianti, A. Pasquazi, “Laser cavity-soliton microcombs”, Nat. Photonics 13, 384-389 (2019).