Frequency noise of laser gyros

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Laser gyros are powerful tools to test the predictions of the general theory of relativity. The precision of a measurement of the rotation rate with a laser gyro is limited by the frequency noise of the beat between two counterpropagating modes of a ring laser. The frequency noise of a single mode of a laser is limited by quantum mechanical constraints because it is related to the maximum precision with which the phase of a coherent state can be measured. If two modes are uncorrelated, the variance of the fluctuations of the difference of the their frequencies is the sum of the variance of the frequency noise of the two modes. If two modes are correlated, this result does not hold any longer. In this paper, we show that there are mechanisms in a laser gyro that are capable to dynamically lock the two modes together without forcing the two modes to the same frequency. The lock of modes decouples the noise of the beat note from the frequency noise of the individual modes, and allows the realization of sub-shot noise laser gyros. These mechanism may explain the recent observation of sub-shot noise performance of the GINGERino laser gyro recently reported in the literature [1].

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Laser gyros are a powerful tool to test general relativity predictions [2–4]. They enable a precise measurement of the rotation rate by measuring the beat of two counterpropagating modes of a ring laser. The basic idea is that rotation breaks the symmetry between conterpropagating modes, and the frequency difference between the two modes is proportional to the rotation rate of the laser gyro. The precision of the measurement depends on the frequency stability of the beat note obtained by detecting the intensity of a coherent combination of the two modes. If the two modes are independent and of equal power, the variance of the frequency noise of the beat is twice the variance of the frequency noise of each individual mode [5–8]. The frequency noise of each individual mode originates from constraints dictated by quantum mechanics and in particular from the precision of a measurement of the phase of a coherent state [5]. The physical mechanisms that make the laser radiation compliant with these constraints come for one half from the quantum noise of the active medium and the other half from the vacuum fluctuations entering from the output port of the laser [9]. Such noise sources are responsible for the phase and frequency noise of the laser, and for the non-zero linewidth of the emitted radiation.

In this paper, we show that a under proper conditions, the two counterpropagating modes of a laser gyro can lock together while still maintaining a different frequency. When these conditions are fulfilled, the noise of the frequency of the beat note decouples to the noise of the individual modes. This result can be understood by the analogy with mode-locked lasers. In passively mode-locked lasers, the locking mechanism is associated to pulsed operation. The linewidth of the single line of the spectrum of the emitted radiation is Lorentzian but the frequency fluctuations are strongly correlated, to the extent that the spectral purity of the beat note between the spectral lines of the emitted frequency comb [10] has been exploited for the realization of very accurate clockworks [11]. In ring lasers, locking of counterpropagating modes is the result of reflections. When reflections occur from static cavity elements like cavity mirrors, the two modes locks at the same frequency. When reflections occurs from the slowly moving grating generated, in a nonlinear medium with slow response, by the beat of the two counterpropagating modes themselves, they tend to stabilize the difference frequency of the two modes. We speculate that this mechanism is at work in the best performing laser gyros operating around the world, when spurious reflections from static cavity elements are minimized, and that may in particular explain the observation of sub-shot noise performance of the GINGERino laser gyro that recently appeared in the literature [1, 12, 13].

One may use these results for investigating the possibility of alternate laser design where a slow saturable absorber is inserted in the laser cavity to stabilize the mode beat. Our findings pave the way for the realization of sub-shot noise laser gyros of unprecedented accuracy for ultra-precise testing of the predictions of general relativity.

1. SINGLE MODE CASE

Following the analysis of Yamamoto and Haus [9], let us consider first a single mode of an empty cavity $\mathbf{a}(t)$ with bosonic commutation relations

$$[\mathbf{a}(t), \mathbf{a}^\dagger(t)] = 1, \tag{1}$$

coupled to an outside optical *wave* $\mathbf{s}_a(t)$ with commutation relations

$$[\mathbf{s}_a(t), \mathbf{s}_a^\dagger(t')] = \delta(t - t'). \tag{2}$$

The wave reflected from the cavity is given by [9, 14]

$$\mathbf{r}_a(t) = -\mathbf{s}_a(t) + \sqrt{\gamma}\mathbf{a}(t). \tag{3}$$

The temporal evolution of the mode $\mathbf{a}(t)$ is described by the differential equation

$$\frac{\mathrm{d}\mathbf{a}(t)}{\mathrm{d}t} = -\frac{\gamma}{2}\mathbf{a}(t) + \sqrt{\gamma}\,\mathbf{s}_a(t). \tag{4}$$

Assuming that the outside wave is incident upon the cavity from a time much longer than $1/\gamma$ solution of Eq. (3) is

$$a(t) = \sqrt{\gamma}\int_{-\infty}^{t} \mathrm{d}u \exp\left[-\frac{\gamma}{2}(t-u)\right]\mathbf{s}_a(u), \tag{5}$$

so that the two-time commutation relations of $\mathbf{r}_a(t)$ are

$$\begin{aligned}[\mathbf{a}(t), \mathbf{a}^\dagger(t')] \quad &= \quad \gamma \exp\left[-\frac{\gamma}{2}(t+t')\right]\int_{-\infty}^{t}\mathrm{d}u\int_{-\infty}^{t'}\mathrm{d}u' \\ &\quad \exp\left[\frac{\gamma}{2}(u+u')\right][\mathbf{s}_a(u), \mathbf{s}_a^\dagger(u')],\end{aligned} \tag{6}$$

that is

$$[\mathbf{a}(t), \mathbf{a}^\dagger(t')] = \exp\left(-\frac{\gamma}{2}|t-t'|\right). \tag{7}$$

consistent with the bosonic commutation rule (1) for $t = t'$.

The commutation relations of the reflected wave $\mathbf{r}_a(t)$ are

$$\begin{aligned}[\mathbf{r}_a(t), \mathbf{r}_a^\dagger(t')] \quad &= \quad [\mathbf{s}_a(t), \mathbf{s}_a^\dagger(t')] + \gamma[\mathbf{a}(t), \mathbf{a}^\dagger(t')] \\ &\quad -\sqrt{\gamma}\left([\mathbf{a}(t), \mathbf{s}_a^\dagger(t')] + [\mathbf{s}_a(t), \mathbf{a}^\dagger(t')]\right).\end{aligned} \tag{8}$$

Being

$$[\mathbf{a}(t), \mathbf{s}_a^\dagger(t')] = \sqrt{\gamma}\int_{-\infty}^{t}\mathrm{d}u \exp\left[-\frac{\gamma}{2}(t-u)\right][\mathbf{s}_a(u), \mathbf{s}_a^\dagger(t')], \tag{9}$$

that is

$$[\mathbf{a}(t), \mathbf{s}_a^\dagger(t')] = \exp\left[-\frac{\gamma}{2}(t-t')\right]u(t-t'), \tag{10}$$

and also

$$[\mathbf{s}_a(t), \mathbf{a}^\dagger(t')] = \exp\left[-\frac{\gamma}{2}(t'-t)\right]u(t'-t), \tag{11}$$

where $u(t) = 1$ for $t > 0$, $u(t) = 0$ for $t < 0$ and $u(0) = 1/2$, so that we obtain

$$[\mathbf{r}_a(t), \mathbf{r}_a^\dagger(t')] = [\mathbf{s}_a(t), \mathbf{s}_a^\dagger(t)], \tag{12}$$

and hence that the commutation relation of the output optical wave are the same of the input wave, as it should be.

Let us now assume that a gain medium is inserted into the cavity (see Fig. 1), which we represent as a statistical mixture of N two-level atoms. Let us define the operators

$$\sigma_- = \sum_{i=1}^{N}\frac{1}{N}(|1\rangle\langle 2|)_i, \tag{13}$$

where $|1\rangle$ and $|2\rangle$ are the two levels, and

$$\sigma_3 = \sum_{i=1}^{N}\frac{1}{N}(|2\rangle\langle 2| - |1\rangle\langle 1|)_i. \tag{14}$$

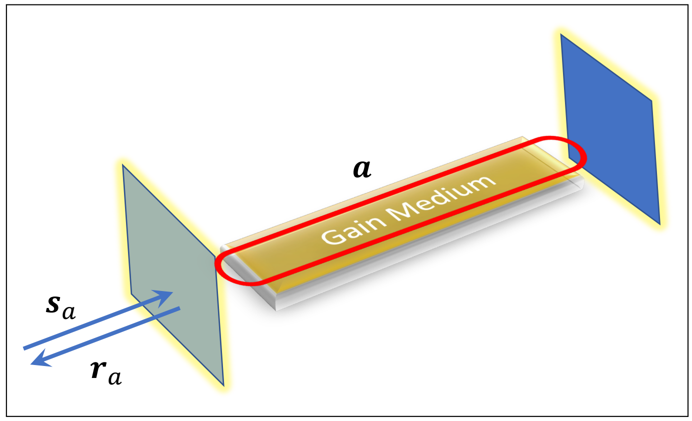

Fig. 1. Representation of the laser cavity. The front mirror is a partially reflecting mirror with power reflectivity R such that $\gamma = (1-R)/\tau_{\mathrm{rt}}$ where τ_{rt} is the cavity roundtrip time, whereas the backward mirror is fully reflecting.

It is easy to show that σ_- and σ_3 obey the commutation relations

$$[\sigma_-, \sigma_-^\dagger] = -\frac{\sigma_3}{N}. \tag{15}$$

and the anti-commutation

$$\{\sigma_-, \sigma_-^\dagger\} = \frac{1}{N}. \tag{16}$$

The spontaneous decay of σ_3 is described by

$$\frac{\mathrm{d}\sigma_-(t)}{\mathrm{d}t} = -\Gamma\sigma_-(t) + \left(\frac{2\Gamma}{N}\right)^{1/2}\mathbf{s}^{(-)}(t), \tag{17}$$

where a noise source $\mathbf{s}^{(-)}(t)$ with commutation relation

$$[\mathbf{s}^{(-)}(t), \mathbf{s}^{(-)\dagger}(t')] = -\sigma_3(t)\,\delta(t-t'), \tag{18}$$

and anti-commutation

$$\{\mathbf{s}^{(-)}(t), \mathbf{s}^{(-)\dagger}(t')\} = \delta(t-t'), \tag{19}$$

is required to preserve the commutation and anti-commutation relations, as it may be verified for the commutator (and similarly for the anti-commutator) by calculating $\mathrm{d}[\sigma_3(t), \sigma_3^\dagger(t)]$ and using that

$$[\mathbf{s}^{(-)}(t)\mathrm{d}t, \mathbf{s}^{(-)\dagger}(t)\mathrm{d}t] = -\sigma_3(t)\mathrm{d}t. \tag{20}$$

Being $\sigma_-(t)^2 = 0$ we also have $\mathbf{s}^{(-)}(t)^2 = \mathbf{s}^{(-)\dagger}(t)^2 = 0$, and this completes the characterization of the noise operator. If the active medium is placed into the cavity that we described above, the coupling with the cavity mode is described by the equation for $\mathbf{a}(t)$

$$\frac{\mathrm{d}\mathbf{a}(t)}{\mathrm{d}t} = -\frac{\gamma}{2}\mathbf{a}(t) - igN\sigma_-(t) + \sqrt{\gamma}\,\mathbf{s}_a(t), \tag{21}$$

and by the equation for σ_-

$$\frac{\mathrm{d}\sigma_-(t)}{\mathrm{d}t} = -\Gamma\sigma_-(t) + ig\sigma_3(t)\mathbf{a}(t) + \left(\frac{2\Gamma}{N}\right)^{1/2}\mathbf{s}^{(-)}(t). \tag{22}$$

In the presence of optical pumping with pumping rate R, the equation for the population inversion $\mathbf{n}(t) = N\sigma_3(t)$ is

$$\frac{\mathrm{d}\mathbf{n}(t)}{\mathrm{d}t} = R - \frac{\mathbf{n}}{\tau} + i2gN\left[\mathbf{a}^\dagger(t)\sigma_-(t) - \sigma_-^\dagger(t)\mathbf{a}(t)\right], \tag{23}$$

where τ is the spontaneous carrier lifetime.

Assuming $\Gamma \gg 1/\tau$, we may neglect in Eq. (22) $\mathrm{d}\sigma_-(t)/\mathrm{d}t$ compared to $-\Gamma\sigma_-(t)$. This procedure yields

$$\sigma_-(t) = i\frac{g}{N\Gamma}\mathbf{n}(t)\mathbf{a}(t) + \left(\frac{2}{N\Gamma}\right)^{1/2}\mathbf{s}^{(-)}(t), \tag{24}$$

and this identity once inserted into the equation for $\mathbf{n}(t)$ permits to adiabatically eliminate $\sigma_-(t)$ in Eqs. (21) and (23), which become

$$\frac{\mathrm{d}\mathbf{a}(t)}{\mathrm{d}t} = \left[-\frac{\gamma}{2} + \frac{g^2}{\Gamma}\mathbf{n}(t)\right]\mathbf{a}(t) - ig\left(\frac{2N}{\Gamma}\right)^{1/2}\mathbf{s}^{(-)}(t) + \sqrt{\gamma}\,\mathbf{s}_a(t), \tag{25}$$

$$\begin{aligned}\frac{\mathrm{d}\mathbf{n}(t)}{\mathrm{d}t} &= R - \frac{\mathbf{n}(t)}{\tau} - \frac{4g^2}{\Gamma}\mathbf{n}(t)\,\mathbf{a}^\dagger(t)\mathbf{a}(t) \\ &\quad + i2g\left(\frac{2N}{\Gamma}\right)^{1/2}\left[\mathbf{a}^\dagger(t)\mathbf{s}^{(-)}(t) - \mathbf{s}^{(-)\dagger}(t)\mathbf{a}(t)\right], \end{aligned} \tag{26}$$

The commutation relations of the noise term in Eq. (25)

$$\mathbf{S}_a(t) = -ig\left(\frac{2N}{\Gamma}\right)^{1/2}\mathbf{s}^{(-)}(t) + \sqrt{\gamma}\,\mathbf{s}_a(t) \tag{27}$$

is

$$[\mathbf{S}_a(t), \mathbf{S}_a^\dagger(t')] = 2\left(\frac{\gamma}{2} - \frac{g^2}{\Gamma}\mathbf{n}\right)\delta(t-t'). \tag{28}$$

Using

$$\mathrm{d}[\mathbf{a}(t), \mathbf{a}^\dagger(t)] = [\mathrm{d}\mathbf{a}(t), \mathbf{a}^\dagger(t)] + [\mathbf{a}(t), \mathrm{d}\mathbf{a}^\dagger(t)] + [\mathrm{d}\mathbf{a}(t), \mathrm{d}\mathbf{a}^\dagger(t)], \tag{29}$$

and

$$[\mathrm{d}\mathbf{a}(t), \mathrm{d}\mathbf{a}^\dagger(t)] = [\mathbf{S}_a(t)\mathrm{d}t, \mathbf{S}_a^\dagger(t)\mathrm{d}t] = 2\left[\frac{\gamma}{2} - \frac{g^2}{\Gamma}\mathbf{n}(t)\right]\mathrm{d}t, \tag{30}$$

we may show that the commutation relations (28) imply $\mathrm{d}[\mathbf{a}(t), \mathbf{a}^\dagger(t)] = 0$, thus ensuring the preservation of the commutation relations for $\mathbf{a}(t)$ also in the presence of the interaction with the gain medium.

Let us now linearize Eqs. (25) and (26) around the steady state by setting

$$\mathbf{a}(t) = a_0 + \delta\mathbf{a}(t), \tag{31}$$
$$\mathbf{n}(t) = n_0 + \delta\mathbf{n}(t), \tag{32}$$

with a_0 and b_0 c-numbers. The commutation relations for $\delta\mathbf{a}(t)$ are equal to the commutation relations for $\mathbf{a}(t)$. The steady state value of the population inversion is

$$n_0 = \frac{\gamma\Gamma}{2g^2}, \tag{33}$$

so that linearization of Eqs. (25) and (26) yields

$$\frac{\mathrm{d}\delta\mathbf{a}(t)}{\mathrm{d}t} = \frac{g^2}{\Gamma}a_0\delta\mathbf{n}(t) - ig\left(\frac{2N}{\Gamma}\right)^{1/2}\mathbf{s}^{(-)}(t) + \sqrt{\gamma}\,\mathbf{s}_a(t), \tag{34}$$

$$\begin{aligned}\frac{\mathrm{d}\delta\mathbf{n}(t)}{\mathrm{d}t} &= -\frac{\delta\mathbf{n}(t)}{\tau} - \frac{4g^2}{\Gamma}a_0^2\delta\mathbf{n}(t) \\ &\quad - \frac{4g^2}{\Gamma}n_0a_0\left[\delta\mathbf{a}(t) + \delta\mathbf{a}^\dagger(t)\right] \\ &\quad + i2g\left(\frac{2N}{\Gamma}\right)^{1/2}a_0\left[\mathbf{s}^{(-)}(t) - \mathbf{s}^{(-)\dagger}(t)\right]. \end{aligned} \tag{35}$$

where we assumed a_0 as real implying the definition of a phase reference for the field.

Adiabatic elimination of the population inversion in the high gain regime in which $1/\tau \ll 4g^2a_0^2/\Gamma$ gives

$$\begin{aligned}\delta\mathbf{n}(t) &= -\frac{n_0}{a_0}\left[\delta\mathbf{a}(t) + \delta\mathbf{a}^\dagger(t)\right] \\ &\quad + i\frac{\Gamma}{2g}\left(\frac{2N}{\Gamma}\right)^{1/2}\frac{1}{a_0}\left[\mathbf{s}^{(-)}(t) - \mathbf{s}^{(-)\dagger}(t)\right]. \end{aligned} \tag{36}$$

This equation, inserted into Eq. (34) gives

$$\begin{aligned}\frac{\mathrm{d}\delta\mathbf{a}(t)}{\mathrm{d}t} &= -\gamma\frac{\delta\mathbf{a}(t) + \delta\mathbf{a}^\dagger(t)}{2} + \sqrt{\gamma}\,\mathbf{s}_a(t) \\ &\quad - ig\left(\frac{2N}{\Gamma}\right)^{1/2}\frac{\mathbf{s}^{(-)}(t) + \mathbf{s}^{(-)\dagger}(t)}{2}. \end{aligned} \tag{37}$$

With strong pumping, the medium is fully inverted so that $n_0 \simeq N$ so that, using Eq. (33) we obtain $\gamma = 2n_0g^2/\Gamma \simeq 2Ng^2/\Gamma$ and therefore

$$\begin{aligned}\frac{\mathrm{d}\delta\mathbf{a}(t)}{\mathrm{d}t} &= -\gamma\frac{\delta\mathbf{a}(t) + \delta\mathbf{a}^\dagger(t)}{2} + \sqrt{\gamma}\,\mathbf{s}_a(t) \\ &\quad - i\sqrt{\gamma}\,\frac{\mathbf{s}^{(-)}(t) + \mathbf{s}^{(-)\dagger}(t)}{2}. \end{aligned} \tag{38}$$

The equations for the in-phase component $\delta\mathbf{a}_1(t) = [\delta\mathbf{a}(t) + \delta\mathbf{a}^\dagger(t)]/2$ and the in-quadrature component $\delta\mathbf{a}_2(t) = [\delta\mathbf{a}(t) - \delta\mathbf{a}^\dagger(t)]/(2i)$ are

$$\frac{\mathrm{d}\delta\mathbf{a}_1(t)}{\mathrm{d}t} = -\gamma\delta\mathbf{a}_1(t) + \sqrt{\gamma}\,\mathbf{s}_{a,1}(t), \tag{39}$$

and

$$\frac{\mathrm{d}\delta\mathbf{a}_2(t)}{\mathrm{d}t} = \sqrt{\gamma}\left[\mathbf{s}_2(t) - \mathbf{s}_1^{(-)}(t)\right], \tag{40}$$

where $\mathbf{s}_1^{(-)}(t) = [\mathbf{s}^{(-)}(t) + \mathbf{s}^{(-)\dagger}(t)]/2$, $\mathbf{s}_2^{(-)}(t) = [\mathbf{s}^{(-)}(t) - \mathbf{s}^{(-)\dagger}(t)]/(2i)$, $\mathbf{s}_{a,1}(t) = [\mathbf{s}_a(t) + \mathbf{s}_a^\dagger(t)]/2$, and $\mathbf{s}_{a,2}(t) = [\mathbf{s}_a(t) - \mathbf{s}_a^\dagger(t)]/(2i)$.

Solving in the Fourier domain the equation for the in-phase component (39) we obtain

$$\delta\mathbf{a}_1(\omega) = \frac{\sqrt{\gamma}\,\mathbf{s}_{a,1}(\omega)}{-i\omega + \gamma}, \tag{41}$$

which inserted into the equation for the fluctuations of $\mathbf{r}_{a,1}(\omega)$ given by Eq. (3) yields

$$\delta\mathbf{r}_{a,1}(\omega) = \frac{i\omega\gamma}{-i\omega + \gamma}\,\mathbf{s}_{a,1}(\omega). \tag{42}$$

For $\omega \ll \gamma$ we have $\delta\mathbf{r}_{a,1}(\omega) \simeq 0$ [9, 14], whereas for $\omega \gg \gamma$ we have $\delta\mathbf{r}_{a,1}(\omega) = -\mathbf{s}_{a,1}(\omega)$, so that in this regime the incoming vacuum fluctuations are reflected from the cavity with a π phase shift, producing a coherent state at output.

Using Eq. (18), and being $\langle\sigma_3\rangle = 1$ for full inversion, we obtain

$$\langle\mathbf{s}_i^{(-)}(t)\mathbf{s}_i^{(-)}(t')\rangle = \frac{1}{4}\delta(t-t'), \quad i = 1, 2, \tag{43}$$

and using Eq. (3)

$$\langle\mathbf{s}_i(t)\mathbf{s}_i(t')\rangle = \frac{1}{4}\delta(t-t'), \quad i = 1, 2. \tag{44}$$

Equation (40) shows that the diffusion coefficient for the in-quadrature fluctuations is equal to $\gamma/2$, so that the diffusion coefficient for the phase fluctuations, defined as

$$\Delta\boldsymbol{\varphi} = \frac{\delta \mathbf{a}_2(t)}{a_0}, \tag{45}$$

is

$$D_\varphi = \frac{\gamma}{2a_0^2}. \tag{46}$$

so that the laser line-width is

$$\Delta\nu = \frac{D_\varphi}{2\pi} = \frac{\gamma}{4\pi a_0^2}. \tag{47}$$

If we use the expression for the output power of the laser $P = \gamma a_0^2 \hbar\omega_0$, we obtain the well-known Schawlow-Townes linewidth formula

$$\Delta\nu = \frac{\gamma^2 \hbar\omega_0}{4\pi P}. \tag{48}$$

The uncertainty of a frequency measurement over a time T is

$$\boldsymbol{\omega}_{\text{meas}} T = \omega_0 T + \Delta\boldsymbol{\varphi}(t+T) - \Delta\boldsymbol{\varphi}(t), \tag{49}$$

so that, using $\langle[\Delta\boldsymbol{\varphi}(t+T) - \Delta\boldsymbol{\varphi}(t)]^2\rangle = D_\varphi T$ we obtain

$$\Delta\omega_{\text{meas}}^2 = \frac{\langle[\Delta\boldsymbol{\varphi}(t+T) - \Delta\boldsymbol{\varphi}(t)]^2\rangle}{T^2} = \frac{\gamma}{2a_0^2 T}, \tag{50}$$

where we defined the uncertainty of the frequency measurement as $\Delta\omega_{\text{meas}} = (\langle\Delta\boldsymbol{\omega}_{\text{meas}}\rangle^2)^{1/2}$. Equation (50) can be interpreted in simple physical terms. The variance of a phase measurement on a coherent state of amplitude a_0 is $\Delta\varphi_{\text{coh}}^2 = 1/(4a_0^2)$. Nyquist criterion states that the number of independent measurements that can be performed over a time T on a signal of correlation time $1/\gamma$ (see Eq. (42)) is $N_{\text{meas}} = (2T)/(1/\gamma)$, so that the variance of the frequency measurement is $\Delta\omega_{\text{coh}}^2 = (\Delta\varphi_{\text{coh}}^2/T^2)/N_{\text{meas}}$, which returns Eq. (50) [5].

Using in Eq. (50), the relation that links a_0^2 to the output power of the laser P, namely $a_0^2 = P/(\gamma\hbar\omega_0)$, we obtain the expression

$$\Delta\omega_{\text{meas}} = \frac{\omega_0}{Q}\sqrt{\frac{\hbar\omega_0}{2PT}}, \tag{51}$$

where we defined the cavity quality factor as $Q = \omega_0/\gamma$.

2. THE LASER GYRO: A TWO-MODE CASE

While the laser linewidth and the precision of a measurement of the frequency of a single laser mode are prone to strong quantum mechanical constraints, the frequency difference of two modes are not. Of course, if two modes are independent the variance of the fluctuations of the difference frequency is the sum of the variances of the individual modes. Different is the case of correlated modes. The case of the beat of two modes of a mode-locked laser is an example where the beat of two modes has a precision orders of magnitude larger than the precision of each individual mode frequency [10]. This property enables the transfer down to microwave frequencies of extremely stable optical oscillations and vice versa [11]. It is therefore worth investigating whether there are any locking mechanisms active (or can be induced by a suitable design) in laser gyros.

Let us consider a ring laser with two counterpropagating modes (see Fig. 2), one forward propagating centered a frequency $\omega_0 + \Omega_0/2$

$$\frac{d\mathbf{a}(t)}{dt} = -i\frac{\Omega_0}{2}\mathbf{a}(t) - \frac{\gamma}{2}\mathbf{a}(t) - igN[\sigma_-(t)]_a + \sqrt{\gamma}\,\mathbf{s}_a(t), \tag{52}$$

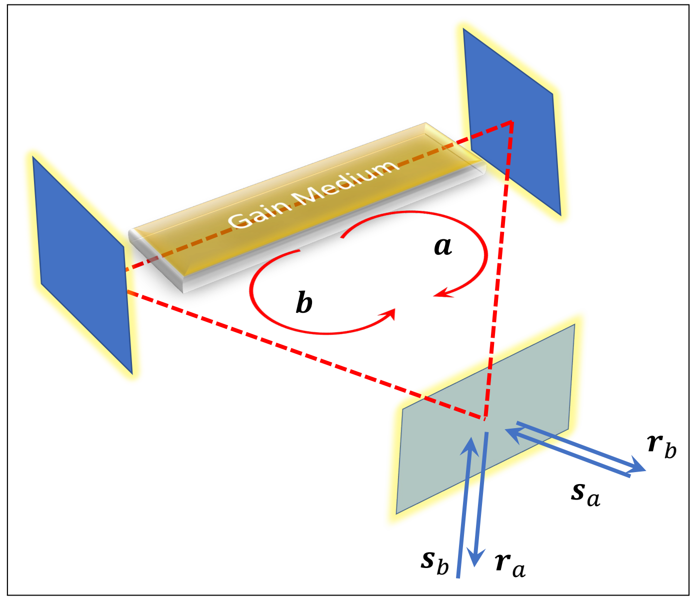

Fig. 2. Representation of the ring laser cavity. The front mirror is a partially reflecting mirror with power reflectivity R such that $\gamma = (1-R)/\tau_{\text{rt}}$ where τ_{rt} is the cavity roundtrip time, whereas other two mirrors are fully reflecting.

and the other backward propagating centered at frequency $\omega_0 - \Omega_0/2$

$$\frac{d\mathbf{b}(t)}{dt} = i\frac{\Omega_0}{2}\mathbf{b}(t) - \frac{\gamma}{2}\mathbf{b}(t) - igN[\sigma_-(t)]_b + \sqrt{\gamma}\,\mathbf{s}_b(t). \tag{53}$$

Here $[\sigma_-(t)]_a$ and $[\sigma_-(t)]_b$ are the (suitably normalized) spatial Fourier components of σ_- proportional to $\exp(ikz)$ and $\exp(-ikz)$ that couple with the forward and backward propagating waves. The equation for σ_- becomes

$$\frac{d\sigma_-(t)}{dt} = -\Gamma\sigma_-(t) + ig\sigma_3(t)(\mathbf{a}(t) + \mathbf{b}(t)) + \left(\frac{2\Gamma}{N}\right)^{1/2}\mathbf{s}^{(-)}(t). \tag{54}$$

In the presence of optical pumping with pumping rate R, the equation for the population inversion $\mathbf{n}(t) = N\sigma_3(t)$ is

$$\begin{aligned}\frac{d\mathbf{n}(t)}{dt} &= R - \frac{\mathbf{n}}{\tau} + i2gN\big[(\mathbf{a}^\dagger(t) + \mathbf{b}^\dagger(t))\sigma_-(t) \\ &\quad -\sigma_-^\dagger(t)(\mathbf{a}(t) + \mathbf{b}(t))\big].\end{aligned} \tag{55}$$

where τ is the spontaneous lifetime.

Adiabatic elimination of $\sigma_-(t)$ in Eq. (54) gives

$$\sigma_-(t) = i\frac{g}{N\Gamma}\mathbf{n}(t)(\mathbf{a}(t) + \mathbf{b}(t)) + \left(\frac{2}{N\Gamma}\right)^{1/2}\mathbf{s}^{(-)}(t), \tag{56}$$

that is, the expected linear dependence of the medium polarization on the optical field. Inserting Eq. (56) into Eqs. (52) and (53) and projecting $\sigma_-(t)$ over the two counterpropagating modes gives

$$\begin{aligned}\frac{d\mathbf{a}(t)}{dt} &= -i\frac{\Omega_0}{2}\mathbf{a}(t) + \left[-\frac{\gamma}{2} + \frac{g^2}{\Gamma}\mathbf{n}(t)\right]\mathbf{a}(t) \\ &\quad -ig\left(\frac{2N}{\Gamma}\right)^{1/2}\mathbf{s}_a^{(-)}(t) + \sqrt{\gamma}\,\mathbf{s}_a(t),\end{aligned} \tag{57}$$

$$\frac{\mathrm{d}\mathbf{b}(t)}{\mathrm{d}t} = i\frac{\Omega_0}{2}\mathbf{b}(t) + \left[-\frac{\gamma}{2} + \frac{g^2}{\Gamma}\mathbf{n}(t)\right]\mathbf{b}(t) - ig\left(\frac{2N}{\Gamma}\right)^{1/2}\mathbf{s}_b^{(-)}(t) + \sqrt{\gamma}\,\mathbf{s}_b(t). \quad (58)$$

Here $\mathbf{s}_{a,b}^{(-)}(t)$ are the result of the projection of the noise term $\mathbf{s}^{(-)}(t)$ over the spatial mode profile $\exp(ikz)$ and $\exp(-ikz)$. Local multiplication by $\exp(\pm ikz)$ generates two independent noise terms with the same commutation properties of $\mathbf{s}^{(-)}(t)$. As a check, it may be verified that, if $\mathbf{s}_{a,b}^{(-)}(t)$ obey the commutation rule (18), Eqs. (57) and (58) preserve the bosonic commutation rules of the two modes. Entering Eq. (56) into Eq. (55) and expanding the product of the mode amplitudes yields

$$\begin{aligned}\frac{\mathrm{d}\mathbf{n}(t)}{\mathrm{d}t} = {} & R - \frac{\mathbf{n}(t)}{\tau} - \frac{4g^2}{\Gamma}\mathbf{n}(t)\left[\mathbf{a}^\dagger(t)\mathbf{a}(t) + \mathbf{b}^\dagger(t)\mathbf{b}(t)\right. \\ & \left. + \mathbf{a}^\dagger(t)\mathbf{b}(t) + \mathbf{b}^\dagger(t)\mathbf{a}(t)\right] \\ & + i2g\left(\frac{2N}{\Gamma}\right)^{1/2}\left[(\mathbf{a}^\dagger(t) + \mathbf{b}^\dagger(t))\mathbf{s}^{(-)}(t)\right. \\ & \left. - \mathbf{s}^{(-)\dagger}(t)(\mathbf{a}(t) + \mathbf{b}(t))\right]. \end{aligned} \quad (59)$$

Being $\Omega \ll 1/\tau$, we may assume that $\mathbf{n}(t)$ adiabatically follows the modulation frequency, so that the steady state of $\mathbf{n}$ is

$$n(t) = \frac{R}{1/\tau + (4g^2/\Gamma)\left(|a_0|^2 + |b_0|^2 + a_0^* b_0 e^{i\Omega_0 t} + a_0 b_0^* e^{-i\Omega_0 t}\right)} \quad (60)$$

The the terms $a_0^* b_0$ and $a_0 b_0^*$ account for a gain grating that is generated by the beat of the two counterpropagating modes over the gain medium. The nature of this grating may be understood by considering that the two counterpropagating modes collide over the active medium and generate the intensity pattern

$$I(z,t) = |A\exp(-i\Omega_0 t/2 + ikz) + B\exp(i\Omega_0 t/2 - ikz)|^2, \quad (61)$$

where A and B are the amplitudes of the forward and backward propagating modes at the position of the gain medium. Expanding the expression of the intensity, we obtain

$$\begin{aligned} I(z,t) = {} & |A|^2 + |B|^2 + AB^*\exp(-i\Omega_0 t + 2ikz) \\ & + A^* B\exp(i\Omega_0 t - 2ikz). \end{aligned} \quad (62)$$

The grating moves at the speed $\Omega_0/(2k) = (f_1 - f_2)\lambda/2$, in the GINGERino case [15, 16] about 89 microns per second.

In a gas laser, the amplitude of the grating tends to be attenuated by diffusion, so that we may expand to first order the above expression

$$n(t) = n_0\left[1 - \xi\frac{a_0^* b_0 e^{i\Omega_0 t} + a_0 b_0^* e^{-i\Omega_0 t}}{1/\tau + (4g^2/\Gamma)\left(|a_0|^2 + |b_0|^2\right)}\right] \quad (63)$$

where

$$n_0 = \frac{R}{1/\tau + (4g^2/\Gamma)\left(|a_0|^2 + |b_0|^2\right)}, \quad (64)$$

and where $\xi < 1$ is a factor accounting for the reduction of the grating amplitude caused by diffusion of the active atoms.

Similarly to the single mode case, the phase fluctuations are independent of the fluctuations of the carrier, so that $\mathbf{n}(t)$ can be replaced by its steady state value n_0

$$\begin{aligned}\frac{\mathrm{d}\mathbf{a}(t)}{\mathrm{d}t} = {} & -i\frac{\Omega_0}{2}\mathbf{a}(t) + \kappa_g\mathbf{b}(t)e^{-i\Omega_0 t} + \left(-\frac{\gamma}{2} + \frac{g^2}{\Gamma}n_0\right)\mathbf{a}(t) \\ & - i\sqrt{\gamma}\,\mathbf{s}_a^{(-)}(t) + \sqrt{\gamma}\,\mathbf{s}_a(t), \end{aligned} \quad (65)$$

$$\begin{aligned}\frac{\mathrm{d}\mathbf{b}(t)}{\mathrm{d}t} = {} & i\frac{\Omega_0}{2}\mathbf{b}(t) + \kappa_g^*\mathbf{a}(t)e^{i\Omega_0 t} + \left(-\frac{\gamma}{2} + \frac{g^2}{\Gamma}n_0\right)\mathbf{b}(t) \\ & - i\sqrt{\gamma}\,\mathbf{s}_b^{(-)}(t) + \sqrt{\gamma}\,\mathbf{s}_b(t). \end{aligned} \quad (66)$$

where we used $\gamma = 2g^2 n_0/\Gamma$ and assumed full inversion so that $N \simeq n_0$, and we defined

$$\kappa_g = -\frac{g^2 n_0}{\Gamma}\frac{\xi\, a_0 b_0^*}{1/\tau + (4g^2/\Gamma)\left(|a_0|^2 + |b_0|^2\right)}. \quad (67)$$

The term κ_g, proportional to $a_0 b_0^*$ couples the backward propagating mode to the forward propagating mode, because the spatial modulation proportional to $\exp(2ikz)$ promotes phase matching between the backward propagating wave, with spatial dependence $\exp(-ikz)$, and the forward propagating wave, with spatial dependence $\exp(ikz)$. By a similar mechanism, the term κ_g^*, proportional to $a_0^* b_0$ couples the forward propagating mode to the backward propagating mode.

Reflections may also occur from various optical elements in the optical cavity, primarily from cavity mirrors. In this case, however, reflections do not change the frequency of the field. Including this process into Eqs. (65) and (66) by an extra backscattering coefficient κ_m, they become

$$\begin{aligned}\frac{\mathrm{d}\mathbf{a}(t)}{\mathrm{d}t} = {} & -i\frac{\Omega_0}{2}\mathbf{a}(t) + \left(\kappa_g e^{-i\Omega_0 t} + \kappa_m\right)\mathbf{b}(t) \\ & + \left(-\frac{\gamma}{2} + \frac{g^2}{\Gamma}n_0\right)\mathbf{a}(t) - i\sqrt{\gamma}\,\mathbf{s}_a^{(-)}(t) + \sqrt{\gamma}\,\mathbf{s}_a(t), \end{aligned} \quad (68)$$

$$\begin{aligned}\frac{\mathrm{d}\mathbf{b}(t)}{\mathrm{d}t} = {} & i\frac{\Omega_0}{2}\mathbf{b}(t) + \left(\kappa_g^* e^{i\Omega_0 t} + \kappa_m^*\right)\mathbf{a}(t) \\ & + \left(-\frac{\gamma}{2} + \frac{g^2}{\Gamma}n_0\right)\mathbf{b}(t) - i\sqrt{\gamma}\,\mathbf{s}_b^{(-)}(t) + \sqrt{\gamma}\,\mathbf{s}_b(t). \end{aligned} \quad (69)$$

These equations are linear in the fields. So, a meaningful analysis can be performed assuming classical fields, with noise sources whose strength are dictated by quantum mechanics. Considering only the deterministic part, and defining $\Delta g = -\gamma + 2g^2 n_0/\Gamma$, $a_0 = |a_0|\exp(i\varphi_a)$, $b_0 = |b_0|\exp(i\varphi_b)$, $\kappa_g = |\kappa_g|\exp(i\varphi_g)$ and $\kappa_g = |\kappa_m|\exp(i\varphi_m)$ we obtain

$$\begin{aligned}\frac{\mathrm{d}\varphi_a}{\mathrm{d}t} = {} & -\frac{\Omega_0}{2} + \frac{|b_0|}{|a_0|}\left[|\kappa_g|\sin(\varphi_b - \varphi_a + \varphi_g - \Omega_0 t)\right. \\ & \left. + |\kappa_m|\sin(\varphi_b - \varphi_a + \varphi_m)\right], \end{aligned} \quad (70)$$

$$\begin{aligned}\frac{\mathrm{d}\varphi_b}{\mathrm{d}t} = {} & \frac{\Omega_0}{2} - \frac{|a_0|}{|b_0|}\left[|\kappa_g|\sin(\varphi_b - \varphi_a + \varphi_g - \Omega_0 t)\right. \\ & \left. + |\kappa_m|\sin(\varphi_b - \varphi_a + \varphi_m)\right]. \end{aligned} \quad (71)$$

We also have

$$\begin{aligned}\frac{\mathrm{d}|a_0|}{\mathrm{d}t} = {} & \frac{\Delta g}{2}|a_0| + |b_0|\left[|\kappa_g|\cos(\varphi_b - \varphi_a + \varphi_g - \Omega_0 t)\right. \\ & \left. + |\kappa_m|\cos(\varphi_b - \varphi_a + \varphi_m)\right], \end{aligned} \quad (72)$$

$$\begin{aligned}\frac{\mathrm{d}|b_0|}{\mathrm{d}t} = {} & \frac{\Delta g}{2}|b_0| + |a_0|\left[|\kappa_g|\cos(\varphi_b - \varphi_a + \varphi_g - \Omega_0 t)\right. \\ & \left. + |\kappa_m|\cos(\varphi_b - \varphi_a + \varphi_m)\right]. \end{aligned} \quad (73)$$

These equations admit stable stationary solutions when either $|\kappa_g|$ or $|\kappa_m|$ is predominant, so that the other can be neglected. Let us consider these two cases separately.

A. Scattering due to mirrors is predominant

This case corresponds to $\kappa_g = 0$. In this case, defining $\Delta\varphi = \varphi_a - \varphi_b - \varphi_m$ we obtain

$$\frac{\mathrm{d}\Delta\varphi}{\mathrm{d}t} = \Omega_0 - |\kappa_m| \left(\frac{|b_0|}{|a_0|} + \frac{|a_0|}{|b_0|} \right) \sin(\Delta\varphi). \quad \textbf{(74)}$$

and also

$$\frac{\mathrm{d}|a_0|}{\mathrm{d}t} = \frac{\Delta g}{2}|a_0| + |\kappa_m||b_0|\cos(\Delta\varphi), \quad \textbf{(75)}$$

$$\frac{\mathrm{d}|b_0|}{\mathrm{d}t} = \frac{\Delta g}{2}|b_0| + |\kappa_m||a_0|\cos(\Delta\varphi). \quad \textbf{(76)}$$

Of course, if $|\kappa_m|$ is negligible $\Delta\varphi = \Omega_0 t$. However, two steady state solutions with a time-independent value of $\Delta\varphi$ exist if $\Omega_0 \leq 2|\kappa_m|$. This steady state corresponds to two counterpropagating modes with the same frequency and locked phase, and is achieved for $|a_0| = |b_0|$, $\Delta g = -2|\kappa_m|\cos(\Delta\varphi)$ and for values of Ω_0 such that

$$\Omega_0 = 2|\kappa_m|\sin(\Delta\varphi). \quad \textbf{(77)}$$

Of the two solutions, only that with $\Delta g = -2|\kappa_m|\cos(\Delta\varphi) < 0$ is stable. The maximum value of Ω_0 compatible with this steady state solution is $\Omega_{\mathrm{lock-in}} = 2|\kappa_m|$.

Locking at a zero difference frequency should be avoided in the proper operation of a laser gyro. The value of $|\kappa_m|$ can be estimated from the frequency $f_{\mathrm{lock-in}} = \Omega_{\mathrm{lock-in}}/(2\pi)$ reported for operating laser gyros in Table II of ref. [4], which ranges from 8 to 240 mHz.

When the locking condition is established, then the two modes of equal frequency produce in the gain medium a static standing grating, and the reflection from this grating further stabilize the locking state. The effect in a laser gyro of reflections from a standing gain and index grating was described in [2].

B. Scattering due to gain is predominant

This case corresponds to set $\kappa_m = 0$ in Eqs. (68) and (69), and is more conveniently studied by frequency shifting $\mathbf{a}(t)$ by $-\Omega_0/2$ and $\mathbf{a}(t)$ by $\Omega_0/2$ by

$$\mathbf{a}'(t) = \mathbf{a}(t)\exp(i\Omega_0 t/2), \quad \textbf{(78)}$$

$$\mathbf{b}'(t) = \mathbf{b}(t)\exp(-i\Omega_0 t/2), \quad \textbf{(79)}$$

so that the transformed field $\mathbf{a}'(t)$ is centered at frequency $\omega_0 - \Omega_0/2$ and $\mathbf{b}'(t)$ around $\omega_0 + \Omega_0/2$. The new fields obey the following equations:

$$\begin{aligned}\frac{\mathrm{d}\mathbf{a}'(t)}{\mathrm{d}t} &= \kappa_g \mathbf{b}'(t) + \left[-\frac{\gamma}{2} + \frac{g^2}{\Gamma}\mathbf{n}(t) \right] \mathbf{a}'(t) \\ &+ \left[-i\sqrt{\gamma}\,\mathbf{s}_a^{(-)}(t) + \sqrt{\gamma}\,\mathbf{s}_a(t) \right] e^{i\Omega_0 t/2}, \end{aligned} \quad \textbf{(80)}$$

$$\begin{aligned}\frac{\mathrm{d}\mathbf{b}'(t)}{\mathrm{d}t} &= \kappa_g^* \mathbf{a}'(t) + \left[-\frac{\gamma}{2} + \frac{g^2}{\Gamma}\mathbf{n}(t) \right] \mathbf{b}'(t) \\ &+ \left[-i\sqrt{\gamma}\,\mathbf{s}_b^{(-)}(t) + \sqrt{\gamma}\,\mathbf{s}_b(t) \right] e^{-i\Omega_0 t/2}. \end{aligned} \quad \textbf{(81)}$$

The transformations (78) and (79) allow us to define independent phase references for the two modes. Defining $\Delta\varphi' = \varphi_a' - \varphi_b' - \varphi_g$, where φ_a' and $-\varphi_b'$ are the phases of the frequency shifted fields, corresponding to $\Delta\varphi' = \varphi_a - \varphi_b - \varphi_g + \Omega_0 t$ in terms of the phases of the original fields, we obtain

$$\frac{\mathrm{d}\Delta\varphi'}{\mathrm{d}t} = -|\kappa_g| \left(\frac{|b_0|}{|a_0|} + \frac{|a_0|}{|b_0|} \right) \sin(\Delta\varphi'). \quad \textbf{(82)}$$

and also

$$\frac{\mathrm{d}|a_0|}{\mathrm{d}t} = \frac{\Delta g}{2}|a_0| + |\kappa_g||b_0|\cos(\Delta\varphi), \quad \textbf{(83)}$$

$$\frac{\mathrm{d}|b_0|}{\mathrm{d}t} = \frac{\Delta g}{2}|b_0| + |\kappa_g||a_0|\cos(\Delta\varphi). \quad \textbf{(84)}$$

Steady state is achieved for $|a_0| = |b_0|$, $\Delta g = -2|\kappa_g|\cos(\Delta\varphi)$ and for values of $\Delta\varphi = 0$ and $\Delta\varphi = \pi$. Of the two solutions, only $\Delta\varphi = 0$ is stable because $\Delta g = -2|\kappa_g|\cos(\Delta\varphi) < 0$. This condition correspond to a locking of the two modes at a difference frequency Ω_0.

Linearization of Eq. (82) about the steady state $\Delta\varphi' = 0$ (and removing the prime for simplicity of notation) gives

$$\frac{\mathrm{d}\Delta\varphi}{\mathrm{d}t} = -2|\kappa|\Delta\varphi. \quad \textbf{(85)}$$

This equation can be extended to the quantum domain defining $\boldsymbol{\Delta\varphi} = \delta\mathbf{a}_2'/a_0 - \delta\mathbf{b}_2'/b_0$ and adding the proper noise terms as

$$\frac{\mathrm{d}\boldsymbol{\Delta\varphi}}{\mathrm{d}t} = -2|\kappa|\boldsymbol{\Delta\varphi} + \mathbf{s}_{\Delta\varphi}, \quad \textbf{(86)}$$

where

$$\begin{aligned}\mathbf{s}_{\Delta\varphi} &= \frac{\sqrt{\gamma}}{a_0}\left[\mathbf{s}_{a,2}^{(-)}(t) + \mathbf{s}_{a,1}(t) \right] e^{i\Omega_0 t/2} \\ &- \frac{\sqrt{\gamma}}{b_0}\left[\mathbf{s}_{b,2}^{(-)}(t) + \mathbf{s}_{b,1}(t) \right] e^{-i\Omega_0 t/2}. \end{aligned} \quad \textbf{(87)}$$

The frequency shift of the two independent white noise terms in the two lines of Eq. (87) has no effect on their statistical properties, and can be neglected. Solution of Eq. (87) shows that $\Delta\varphi$ has a Lorentzian spectrum. The phase noise $\boldsymbol{\Delta\varphi}$ is a stationary process with power spectrum

$$W_{\Delta\varphi}(\omega) = \frac{\gamma^2\hbar\omega_0}{P(\omega^2 + 4|\kappa_g|^2)}, \quad \textbf{(88)}$$

corresponding to the following auto-correlation function of the phase fluctuations

$$\langle \Delta\varphi(t+\tau)\Delta\varphi(t) \rangle = \frac{\gamma^2\hbar\omega_0}{4P|\kappa_g|}\exp\left(-2|\kappa_g||\tau|\right). \quad \textbf{(89)}$$

Here, we assumed once again full inversion $\langle\sigma_3\rangle = 1$. The power spectrum of the (angular) frequency fluctuations is therefore

$$W_{\Delta\omega_{\mathrm{meas}}}(\omega) = \frac{\gamma^2\hbar\omega_0}{P}\frac{\omega^2}{\omega^2 + 4|\kappa_g|^2}. \quad \textbf{(90)}$$

The uncertainty of a frequency measurement performed over a time T is

$$\boldsymbol{\omega}_{\mathrm{meas}}T = \omega_0 T + \boldsymbol{\Delta\varphi}(t+T) - \boldsymbol{\Delta\varphi}(t), \quad \textbf{(91)}$$

so that, using that $\langle\Delta\varphi^2\rangle = 2\langle\varphi(t)\rangle - 2\langle\Delta\varphi(t+T)\Delta\varphi(t)\rangle$, the uncertainty in a frequency measurement defined like in Eq. (50), is

$$\Delta\omega_{\mathrm{meas}}^2 = \frac{\gamma^2\hbar\omega_0}{2PT^2|\kappa_g|}\left[1 - \exp\left(-2|\kappa_g|T\right)\right], \quad \textbf{(92)}$$

that is, using $\gamma = \omega_0/Q$,

$$\Delta\omega_{\mathrm{meas}} = \frac{\omega_0}{Q}\sqrt{\frac{\hbar\omega_0}{PT}\left[\frac{1 - \exp\left(-2|\kappa_g|T\right)}{2|\kappa_g|T}\right]}. \quad \textbf{(93)}$$

In the limit $|\kappa_g|T \to 0$ we obtain

$$\Delta\omega_{\text{meas}} = \frac{\omega_0}{Q}\sqrt{\frac{\hbar\omega_0}{PT}}, \quad |\kappa_g|T \to 0, \tag{94}$$

that is, the known result for independent modes and $\sqrt{2}$ times bigger than the frequency uncertainty of a single mode given by Eq. (51) [5], whereas for $|\kappa_g|T \gg 1$ we have

$$\Delta\omega_{\text{meas}} = \frac{\omega_0}{QT}\sqrt{\frac{\hbar\omega_0}{2P|\kappa_g|}}, \quad |\kappa_g|T \gg 1. \tag{95}$$

The Allan variance can be easily calculated from the autocorrelation function as

$$\sigma_T^2 = \frac{\gamma^2\hbar\omega_0}{4P|\kappa_g|T^2}\left[3 - 4\exp\left(-2|\kappa_g|T\right) + \exp\left(-4|\kappa_g|T\right)\right]. \tag{96}$$

For $|\kappa_g|T \to 0$ we obtain the Allan variance for unlocked modes, corresponding to white frequency noise

$$\sigma_T^2 = \frac{\gamma^2\hbar\omega_0}{PT}, \quad |\kappa_g|T \to 0, \tag{97}$$

whereas for $|\kappa_g|T \gg 1$ the Allan variance of white phase noise

$$\sigma_T^2 = \frac{3\gamma^2\hbar\omega_0}{4P|\kappa_g|T^2}, \quad |\kappa_g|T \gg 1. \tag{98}$$

3. CONCLUSIONS

In the absence of locking, the two modes fluctuate independently and their phase difference undertakes free diffusion. The effect of the gain grating is to lock the relative phase of the two modes. While free diffusion of the individual modes is not affected, the relative phase diffusion is suppressed. Mathematically, this is the result of the appearance of a restoring force in the dynamical equation for the phase difference. This effectively suppresses the effect of the quantum noise on the phase difference between the two modes, stabilizing the difference frequency of the laser gyro.

This scenario is very similar to the mode-locked laser case [10], where the linewidths of the individual lines of the frequency comb have a Lorentzian shape with the Schawlow-Townes linewidth corresponding to the total intracavity power, whereas the linewidth of the beat is delta-like if repetition rate of the laser is locked to an external microwave source by a feedback loop acting upon the cavity length [10]. This property is used in the realization of clockworks based on optical transitions using phase stabilized mode-locked lasers [11].

In a conventional laser, the mode spacing is determined by the cavity geometry, namely by the roundtrip time. In a laser gyro, the spacing between the two couterpropagating modes is determined by the cavity geometry and by the rotation rate of the gyro, which produces an effective roundtrip time difference between the two modes. In the absence of locking, in both cases, the instantaneous frequency difference between two modes is affected by the independent phase diffusion of the two modes.

The modes of a laser may lock together when the locked configuration requires lower energy than the unlocked one. This is the case of passively mode-locked lasers, where the locked configuration corresponds to a pulsed operation, with pulses energetically preferred because of the presence of a saturable absorbing action within the laser cavity. In the case of a laser gyro where reflection from a dynamical gain (or index) grating occurs, the configuration in which the two modes are locked requires less gain because of the constructive interference with the component of the opposite propagating mode reflected from the gain grating. In the case of mode-locked lasers, the mode beat has a residual linewidth because frequency noise, also originated by the spontaneous emission and hence of quantum origin, couples to the pulse timing via the intracavity dispersion, inducing a timing jitter that perturbs the ideal periodicity of the pulse train [10]. If timing jitter is controlled, like in the case of active mode locking, the individual lines of the frequency comb have a linewidth that depends on the stability of the intracavity optical modulator.

In laser gyros where spurious reflections from the mirrors are minimized, dynamic locking of the two counterpropagating modes is caused by a dynamic gain grating that control the fast fluctuations induced by the spontaneous emission. Like in passively mode-locked lasers, the locking does not prevent the possibility that the mode beat follows the dynamic change of the mode spacing, if this change occurs over a time scale longer than the lifetime of the grating, which is related to the excited state lifetime of the active medium. The locking mechanism may be responsible for the recently observed sub-shot-noise performance of the GINGERino laser gyro [1, 12, 13]. We may speculate that locking of non-degenerate modes may also be stabilized by a suitable design of the laser, adding for instance a slow saturable absorber into the laser cavity, or by a feedback loop with a long integration time acting upon the cavity roundtrip time to stabilize the beat frequency between the two counterpropagating modes.

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Supplementary material: Frequency noise of laser gyros

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This note provides additional information to supplement the study of Ref. [1]. Specifically, it presents a comprehensive derivation of the power spectra for the amplitude and phase fluctuations of the output radiation emitted by the two counter-propagating modes of a laser gyro in the phase-locked regime. The derivation involves solving the linearized equations for the quantum operators that describe the laser dynamics, supplemented with the appropriate quantum noise terms.

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In this note, we give a detailed derivation of the results of [1], by solving the linearized equations describing the two counterpropagating modes for the operators that characterize the laser dynamics, which include the noise operators required to preserve the commutation relations. We will consider only the situation in which locking is caused by the coupling induced by the back-reflection from the gain medium and occurs with a difference frequency Ω_0 between the two modes. We will give the expression for the spectra of the phase and the amplitude of the two modes and of their correlations when the laser operates in this regime.

We will use the annihilation operators $\mathbf{a}(t)$ and $\mathbf{b}(t)$ to represent the amplitudes of the two modes centered at frequency $\omega_0 - \Omega_0/2$ and $\omega_0 + \Omega_0/2$, where ω_0 is the optical frequency. These modes correspond to the primed operators $\mathbf{a}'(t)$ and $\mathbf{b}'(t)$ used in the main text. In addition, we redefine $\mathbf{s}_a(t)e^{i\Omega_0 t/2} \mapsto \mathbf{s}_a(t)$, $\mathbf{s}_b(t)e^{-i\Omega_0 t/2} \mapsto \mathbf{s}_b(t)$, $\mathbf{s}_a^{(-)}(t)e^{i\Omega_0 t/2} \mapsto \mathbf{s}_a^{(-)}(t)$ and $\mathbf{s}_b^{(-)}(t)e^{-i\Omega_0 t/2} \mapsto \mathbf{s}_b^{(-)}(t)$, with the new noise terms having the same statistical properties of the original terms. The equations for the amplitude of the two modes are then [1]

$$\begin{aligned} \frac{\mathrm{d}\mathbf{a}(t)}{\mathrm{d}t} &= \kappa_g \mathbf{b}(t) + \left[-\frac{\gamma}{2} + \frac{g^2}{\Gamma}\mathbf{n}(t)\right]\mathbf{a}(t) \\ &\quad -ig\left(\frac{2N}{\Gamma}\right)^{1/2}\mathbf{s}_a^{(-)}(t) + \sqrt{\gamma}\,\mathbf{s}_a(t), \end{aligned} \tag{1}$$

$$\begin{aligned} \frac{\mathrm{d}\mathbf{b}(t)}{\mathrm{d}t} &= \kappa_g^* \mathbf{a}(t) + \left[-\frac{\gamma}{2} + \frac{g^2}{\Gamma}\mathbf{n}(t)\right]\mathbf{b}(t) \\ &\quad -ig\left(\frac{2N}{\Gamma}\right)^{1/2}\mathbf{s}_b^{(-)}(t) + \sqrt{\gamma}\,\mathbf{s}_b(t). \end{aligned} \tag{2}$$

These equations can be simplified by setting $\kappa_g = |\kappa_g|e^{i\varphi_g}$ and defining $\mathbf{b}(t) = \mathbf{b}'(t)e^{-i\varphi_g/2}$ and $\mathbf{a}(t) = \mathbf{a}'(t)e^{i\varphi_g/2}$. In terms of the new phase shifted fields, the coupling coefficient is real and positive, so that Eqs. (1) and (2) can be rewritten, dropping the primes for convenience of notation, as

$$\begin{aligned} \frac{\mathrm{d}\mathbf{a}(t)}{\mathrm{d}t} &= \kappa_g \mathbf{b}(t) + \left[-\frac{\gamma}{2} + \frac{g^2}{\Gamma}\mathbf{n}(t)\right]\mathbf{a}(t) \\ &\quad -ig\left(\frac{2N}{\Gamma}\right)^{1/2}\mathbf{s}_a^{(-)}(t) + \sqrt{\gamma}\,\mathbf{s}_a(t), \end{aligned} \tag{3}$$

$$\begin{aligned} \frac{\mathrm{d}\mathbf{b}(t)}{\mathrm{d}t} &= \kappa_g \mathbf{a}(t) + \left[-\frac{\gamma}{2} + \frac{g^2}{\Gamma}\mathbf{n}(t)\right]\mathbf{b}(t) \\ &\quad -ig\left(\frac{2N}{\Gamma}\right)^{1/2}\mathbf{s}_b^{(-)}(t) + \sqrt{\gamma}\,\mathbf{s}_b(t). \end{aligned} \tag{4}$$

The coupling of the two modes through a gain grating is non hermitian. As a consequence, the spatial components of the material polarization that couple with the two counterpropagating modes mix, resulting in a statistical dependence of the corresponding noise terms $\mathbf{s}_a^{(-)}$ and $\mathbf{s}_b^{(-)}$, which in absence of coupling are independent. The mixing of the noise terms can be quantified if we require the preservation of the commutation rules $[\mathbf{a}, \mathbf{b}^\dagger] = 0$, which signify the independence of the two modes. This requirement is satisfied if

$$\frac{2Ng^2}{\Gamma}[\mathbf{s}_a^{(-)}(t), \mathbf{s}_b^{(-)\dagger}(t')] = -2\kappa_g \delta(t-t'). \tag{5}$$

The commutation relations alone do not specify the correlations of the noise operators. However, we notice that the noise sources of the material polarization are creation operators so that $\mathbf{s}_a^{(-)}$ and $\mathbf{s}_b^{(-)}$ when applied on the left, and $\mathbf{s}_a^{(-)\dagger}$ and $\mathbf{s}_b^{(-)\dagger}$ when applied on the right, to the state of a fully inverted gain medium should give zero. These conditions, combined with the commutation relations (5), give

$$\langle \mathbf{s}_a^{(-)}(t)\mathbf{s}_b^{(-)\dagger}(t')\rangle = 0, \tag{6}$$

$$\langle \mathbf{s}_b^{(-)\dagger}(t')\mathbf{s}_a^{(-)}(t)\rangle = \frac{\kappa_g}{Ng^2/\Gamma}\delta(t-t'), \tag{7}$$

$$\langle \mathbf{s}_a^{(-)}(t)\mathbf{s}_b^{(-)}(t')\rangle = 0, \tag{8}$$

$$\langle \mathbf{s}_a^{(-)\dagger}(t)\mathbf{s}_b^{(-)\dagger}(t')\rangle = 0. \quad (9)$$

The equation for the carrier number is [1]

$$\begin{aligned}\frac{\mathrm{d}\mathbf{n}(t)}{\mathrm{d}t} &= R - \frac{\mathbf{n}(t)}{\tau} - \frac{4g^2}{\Gamma}\mathbf{n}(t)\left[\mathbf{a}^\dagger(t)\mathbf{a}(t) + \mathbf{b}^\dagger(t)\mathbf{b}(t)\right] \\ &+ i2g\left(\frac{2N}{\Gamma}\right)^{1/2}\left[\mathbf{a}^\dagger(t)\mathbf{s}_a^{(-)}(t) + \mathbf{b}^\dagger(t)\mathbf{s}_b^{(-)}(t)\right. \\ &\left.-\mathbf{s}_a^{(-)\dagger}(t)\mathbf{a}(t) - \mathbf{s}_b^{(-)\dagger}(t)\mathbf{b}(t)\right]. \end{aligned} \quad (10)$$

We did not include in the equation for the carrier number the term

$$\frac{\mathrm{d}\Delta\mathbf{n}(t)}{\mathrm{d}t} = -\frac{4g^2}{\Gamma}\mathbf{n}(t)\left[\mathbf{a}^\dagger(t)\mathbf{b}(t)e^{i\Omega_0 t} + \mathbf{b}^\dagger(t)\mathbf{a}(t)e^{-i\Omega_0 t}\right], \quad (11)$$

responsible for the coupling between the two modes because $\Delta\mathbf{n}(t)$ has been implicitly considered in Eqs. (1) and (2) through the coefficient κ_g and, in the analysis that follows, we will take into consideration the saturation of this term and its influence on the laser dynamics with arguments based on conservation laws.

If we define

$$\Delta g = -\gamma + \frac{2g^2}{\Gamma}n_0, \quad (12)$$

where we have set $\mathbf{n} = n_0 + \delta\mathbf{n}$ with n_0 is the steady state value of $\mathbf{n}$, the condition for steady state of Eqs. (3) and (4) is

$$\begin{aligned}\kappa_g|b_0|e^{i\Delta\varphi} + \frac{\Delta g}{2}|a_0| &= 0, \quad (13)\\ \kappa_g|a_0|e^{-i\Delta\varphi} + \frac{\Delta g}{2}|b_0| &= 0, \quad (14)\end{aligned}$$

where we defined $a_0 = |a_0|e^{i\varphi_a}$, $b_0 = |b_0|e^{i\varphi_b}$, and $\Delta\varphi = \varphi_a - \varphi_b$. Steady state is achieved for the two modes with equal amplitudes $|a_0| = |b_0|$, for $\kappa_g \sin(\Delta\varphi) = 0$ and for $\Delta g = -2\kappa_g\cos(\Delta\varphi)$. Of the two possible solutions $\Delta\varphi = 0$ and $\Delta\varphi = \pi$, only the one with $\Delta g < 0$ is stable. In the following, we will assume without loss of generality that the phase reference for the two modes is chosen such that a_0 and b_0 are real, so that $\varphi_a = \varphi_b = 0$.

Using $\gamma = 2g^2 n_0/\Gamma - \Delta g$ and assuming full inversion $n_0 \simeq N$ and that at steady state $\kappa_g = -\Delta g/2$, and defining $\mathbf{a} = a_0 + \delta\mathbf{a}$ and $\mathbf{b} = b_0 + \delta\mathbf{b}$, the equations for the displacements of the mode amplitudes become

$$\begin{aligned}\frac{\mathrm{d}\delta\mathbf{a}(t)}{\mathrm{d}t} &= \kappa_g\delta\mathbf{b}(t) + \frac{\Delta g}{2}\delta\mathbf{a}(t) + \frac{g^2}{\Gamma}a_0\delta\mathbf{n}(t) \\ &-i\sqrt{\gamma - 2\kappa_g}\,\mathbf{s}_a^{(-)}(t) + \sqrt{\gamma}\,\mathbf{s}_a(t), \quad (15)\\ \frac{\mathrm{d}\delta\mathbf{b}(t)}{\mathrm{d}t} &= \kappa_g\delta\mathbf{a}(t) + \frac{\Delta g}{2}\delta\mathbf{b}(t) + \frac{g^2}{\Gamma}b_0\delta\mathbf{n}(t) \\ &-i\sqrt{\gamma - 2\kappa_g}\,\mathbf{s}_b^{(-)}(t) + \sqrt{\gamma}\,\mathbf{s}_b(t). \quad (16)\end{aligned}$$

The correlation of the noise sources for the gain material are given by Eqs. (6)–(9) where, using $2g^2N/\Gamma = \gamma + \Delta g = \gamma - 2\kappa_g$, Eq. (7) becomes

$$\langle \mathbf{s}_b^{(-)\dagger}(t)\mathbf{s}_a^{(-)}(t')\rangle = \frac{2\kappa_g}{\gamma - 2\kappa_g}\delta(t - t'). \quad (17)$$

Using that at steady state $a_0 = b_0$ and defining the two uncoupled eigenmodes of the system (also known as supermodes)

$$\begin{aligned}\mathbf{c}_+(t) &= \frac{\delta\mathbf{a}(t) + \delta\mathbf{b}(t)}{\sqrt{2}}, \quad (18)\\ \mathbf{c}_-(t) &= \frac{\delta\mathbf{a}(t) - \delta\mathbf{b}(t)}{\sqrt{2}}, \quad (19)\end{aligned}$$

we obtain

$$\begin{aligned}\frac{\mathrm{d}\mathbf{c}_+(t)}{\mathrm{d}t} &= \frac{g^2}{\sqrt{2}\,\Gamma}(a_0 + b_0)\,\delta\mathbf{n}(t) \\ &+\sqrt{\gamma/2}\,[\mathbf{s}_a(t) + \mathbf{s}_b(t)] \\ &-i\sqrt{(\gamma - 2\kappa_g)/2}\left[\mathbf{s}_a^{(-)}(t) + \mathbf{s}_b^{(-)}(t)\right], \quad (20)\\ \frac{\mathrm{d}\mathbf{c}_-(t)}{\mathrm{d}t} &= -2\kappa_g\mathbf{c}_-(t) + \sqrt{\gamma/2}\,[\mathbf{s}_a(t) - \mathbf{s}_b(t)] \\ &-i\sqrt{(\gamma - 2\kappa_g)/2}\left[\mathbf{s}_a^{(-)}(t) - \mathbf{s}_b^{(-)}(t)\right]. \quad (21)\end{aligned}$$

If we define now the noise operators

$$\begin{aligned}\mathbf{s}_+(t) &= \frac{\mathbf{s}_a(t) + \mathbf{s}_b(t)}{\sqrt{2}}, \quad (22)\\ \mathbf{s}_-(t) &= \frac{\mathbf{s}_a(t) - \mathbf{s}_b(t)}{\sqrt{2}}, \\ \mathbf{s}_+^{(-)}(t) &= \sqrt{\frac{\gamma - 2\kappa_g}{\gamma}}\,\frac{\mathbf{s}_a^{(-)}(t) + \mathbf{s}_b^{(-)}(t)}{\sqrt{2}}, \quad (23)\\ \mathbf{s}_-^{(-)}(t) &= \sqrt{\frac{\gamma - 2\kappa_g}{\gamma - 4\kappa_g}}\,\frac{\mathbf{s}_a^{(-)}(t) - \mathbf{s}_b^{(-)}(t)}{\sqrt{2}}, \quad (24)\end{aligned}$$

and use $2g^2 n_0/\Gamma = \gamma - 2\kappa_g$ the above equations become

$$\begin{aligned}\frac{\mathrm{d}\mathbf{c}_+(t)}{\mathrm{d}t} &= \frac{1}{2}(\gamma - 2\kappa_g)\,\frac{a_0 + b_0}{\sqrt{2}}\,\frac{\delta\mathbf{n}(t)}{n_0} \\ &+\sqrt{\gamma}\left[-i\mathbf{s}_+^{(-)}(t) + \mathbf{s}_+(t)\right], \quad (25)\\ \frac{\mathrm{d}\mathbf{c}_-(t)}{\mathrm{d}t} &= -2\kappa_g\mathbf{c}_-(t) - i\sqrt{\gamma - 4\kappa_g}\,\mathbf{s}_-^{(-)}(t) + \sqrt{\gamma}\,\mathbf{s}_-(t). \quad (26)\end{aligned}$$

It may easily be verified that the new noise operators are independent

$$\mathbf{s}_\pm^{(-)}(t)\mathbf{s}_\mp^{(-)}(t) = 0, \quad (27)$$

$$\mathbf{s}_\pm(t)\mathbf{s}_\mp^{(-)\dagger}(t) = \mathbf{s}_\pm^{(-)\dagger}(t)\mathbf{s}_\mp(t) = 0, \quad (28)$$

and have, for full inversion $\langle\sigma_3\rangle = 1$ and $\gamma > 2\kappa_g$, the same commutation relations of the equivalent uncoupled operators,

$$\mathbf{s}_\pm^{(-)\dagger}(t)\mathbf{s}_\pm^{(-)}(t') = \delta(t - t'), \quad (29)$$

$$\mathbf{s}_\pm^{(-)}(t)\mathbf{s}_\pm^{(-)\dagger}(t) = \mathbf{s}_\pm^{(-)}(t)\mathbf{s}_\pm^{(-)}(t) = \mathbf{s}_\pm^{(-)\dagger}(t)\mathbf{s}_\pm^{(-)\dagger}(t) = 0. \quad (30)$$

The commutation relations of $\mathbf{c}_\pm(t)$ are $[\mathbf{c}_\pm(t), \mathbf{c}_\pm^\dagger(t)] = 1$. It is easy to verify that $\mathrm{d}[\mathbf{c}_\pm(t), \mathbf{c}_\pm^\dagger(t)] = 0$ so that Eq. (25) and (26) preserve the commutation relations. Defining the two quadratures for a generic operator $\mathbf{c}_1 = (\delta\mathbf{c} + \delta\mathbf{c}^\dagger)/2$ and $\mathbf{c}_2 = (\delta\mathbf{c} - \delta\mathbf{c}^\dagger)/(2i)$, we obtain

$$\begin{aligned}\frac{\mathrm{d}\mathbf{c}_{-,2}(t)}{\mathrm{d}t} &= -2\kappa_g\mathbf{c}_{-,2}(t) + \sqrt{\gamma}\mathbf{s}_{-,2}(t) - \sqrt{\gamma - 4\kappa_g}\mathbf{s}_{-,1}^{(-)}(t), \quad (31)\\ \frac{\mathrm{d}\mathbf{c}_{-,1}(t)}{\mathrm{d}t} &= -2\kappa_g\mathbf{c}_{-,1}(t) + \sqrt{\gamma}\mathbf{s}_{-,1}(t) + \sqrt{\gamma - 4\kappa_g}\mathbf{s}_{-,2}^{(-)}(t). \quad (32)\end{aligned}$$

The equation for the fluctuations of the carriers is

$$\begin{aligned}\frac{\mathrm{d}\delta\mathbf{n}(t)}{\mathrm{d}t} &= -\frac{\delta\mathbf{n}(t)}{\tau}-\frac{4g^2}{\Gamma}(a_0^2+b_0^2)\delta\mathbf{n}(t)\\ &-\frac{8g^2}{\Gamma}n_0\left[a_0\delta\mathbf{a}_1(t)+b_0\delta\mathbf{b}_1(t)\right]\\ &-4g\left(\frac{2N}{\Gamma}\right)^{1/2}\left[a_0\mathbf{s}_{a,2}^{(-)}(t)+b_0\mathbf{s}_{b,2}^{(-)}(t)\right].\end{aligned} \quad (33)$$

This equation however does not include the effect of the depletion of the carriers that generate the gain grating, whose dynamics is described by Eq. (11). Instead of constructing a model to describe the formation of the gain grating and its interaction with the two counterpropagating modes, which would necessitate making assumptions about the complex physics of the laser that are challenging to evaluate, like for instance the carrier diffusion attenuating the grating amplitude, we choose to introduce a term that account for this effect without a formal derivation, relying on the principle that each photon is generated through the decay of one carrier. To this aim, we notice that the coupling induced by the gain grating produces a rate of photon production

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{a}^\dagger\mathbf{a}+\mathbf{b}^\dagger\mathbf{b}\right)_{\text{coupling}}=2\kappa_g\left(\mathbf{b}^\dagger\mathbf{a}+\mathbf{a}^\dagger\mathbf{b}\right), \quad (34)$$

and therefore the change of carrier number caused by fluctuations of **a** and **b** is

$$\left(\frac{\mathrm{d}\delta\mathbf{n}}{\mathrm{d}t}\right)_{\text{coupling}}=-4\kappa_g\left[a_0\left(\delta\mathbf{b}+\delta\mathbf{b}^\dagger\right)+b_0\left(\delta\mathbf{a}+\delta\mathbf{a}^\dagger\right)\right], \quad (35)$$

where we used that a_0 and b_0 are real. Equation (33) supplemented with the coupling the term (35) becomes

$$\begin{aligned}\frac{\mathrm{d}\delta\mathbf{n}(t)}{\mathrm{d}t} &= -\frac{\delta\mathbf{n}(t)}{\tau}-\frac{4g^2}{\Gamma}(a_0^2+b_0^2)\delta\mathbf{n}(t)\\ &-\frac{8g^2}{\Gamma}n_0\left[a_0\delta\mathbf{a}_1(t)+b_0\delta\mathbf{b}_1(t)\right]\\ &-8\kappa_g\left[a_0\delta\mathbf{b}_1(t)+b_0\delta\mathbf{a}_1(t)\right]\\ &-4g\left(\frac{2N}{\Gamma}\right)^{1/2}\left[a_0\mathbf{s}_{a,2}^{(-)}(t)+b_0\mathbf{s}_{b,2}^{(-)}(t)\right].\end{aligned} \quad (36)$$

Using now once again our assumption of full inversion $n_0=N$, we can replace $2g^2N/\Gamma=\gamma-2\kappa_g$ and therefore

$$\begin{aligned}\frac{\mathrm{d}\delta\mathbf{n}(t)}{\mathrm{d}t} &= -\frac{\delta\mathbf{n}(t)}{\tau}-2(\gamma-2\kappa_g)(a_0^2+b_0^2)\frac{\delta\mathbf{n}(t)}{n_0}\\ &-4(\gamma-2\kappa_g)\left[a_0\delta\mathbf{a}_1(t)+b_0\delta\mathbf{b}_1(t)\right]\\ &-8\kappa_g\left[a_0\delta\mathbf{b}_1(t)+b_0\delta\mathbf{a}_1(t)\right]\\ &-4\sqrt{\gamma-2\kappa_g}\left[a_0\mathbf{s}_{a,2}^{(-)}(t)+b_0\mathbf{s}_{b,2}^{(-)}(t)\right].\end{aligned} \quad (37)$$

Assuming strong saturation and neglecting spontaneous emission compared to stimulated emission, we may assume that the carriers adiabatically follow the field fluctuations, so that we obtain

$$\begin{aligned}\frac{\delta\mathbf{n}(t)}{n_0} &= -\frac{2}{a_0^2+b_0^2}\left[a_0\delta\mathbf{a}_1(t)+b_0\delta\mathbf{b}_1(t)\right]\\ &-\frac{4\kappa_g}{(\gamma-2\kappa_g)(a_0^2+b_0^2)}\left[b_0\delta\mathbf{a}_1(t)+a_0\delta\mathbf{b}_1(t)\right]\\ &-\frac{2}{\sqrt{\gamma-2\kappa_g}(a_0^2+b_0^2)}\left[a_0\mathbf{s}_{a,2}^{(-)}(t)+b_0\mathbf{s}_{b,2}^{(-)}(t)\right].\end{aligned} \quad (38)$$

Entering this expression into Eq. (25) yields

$$\begin{aligned}\frac{\mathrm{d}\mathbf{c}_+(t)}{\mathrm{d}t} &= -\frac{(\gamma-2\kappa_g)(a_0+b_0)}{\sqrt{2}(a_0^2+b_0^2)}\left[a_0\delta\mathbf{a}_1(t)+b_0\delta\mathbf{b}_1(t)\right]\\ &-\frac{2\kappa_g(a_0+b_0)}{\sqrt{2}(a_0^2+b_0^2)}\left[b_0\delta\mathbf{a}_1(t)+a_0\delta\mathbf{b}_1(t)\right]\\ &-\frac{\sqrt{(\gamma-2\kappa_g)}(a_0+b_0)}{\sqrt{2}(a_0^2+b_0^2)}\left[a_0\mathbf{s}_{a,2}^{(-)}(t)+b_0\mathbf{s}_{b,2}^{(-)}(t)\right]\\ &-i\sqrt{\gamma}\mathbf{s}_+^{(-)}(t)+\sqrt{\gamma}\,\mathbf{s}_+(t).\end{aligned} \quad (39)$$

Using the steady state condition $a_0=b_0$ we obtain

$$\begin{aligned}\frac{\mathrm{d}\mathbf{c}_+(t)}{\mathrm{d}t} &= -\gamma\mathbf{c}_{+,1}(t)\\ &-\sqrt{\gamma}\left[\mathbf{s}_{+,2}^{(-)}(t)+i\mathbf{s}_+^{(-)}(t)-\mathbf{s}_+(t)\right].\end{aligned} \quad (40)$$

The in-phase and in-quadrature components obey the equations

$$\frac{\mathrm{d}\mathbf{c}_{+,1}(t)}{\mathrm{d}t} = -\gamma\mathbf{c}_{+,1}(t)+\sqrt{\gamma}\,\mathbf{s}_{+,1}(t), \quad (41)$$

$$\frac{\mathrm{d}\mathbf{c}_{+,2}}{\mathrm{d}t} = \sqrt{\gamma}\left[\mathbf{s}_{+,2}(t)-\mathbf{s}_{+,1}^{(-)}(t)\right]. \quad (42)$$

Defining the Fourier transform as

$$\mathbf{c}(\omega)=\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\exp(-i\omega t)\mathbf{c}(t), \quad (43)$$

we may readily solve Eqs. (31), (32), (41) and (42) in the Fourier domain as

$$\mathbf{c}_{-,1}(\omega) = \frac{\sqrt{\gamma}\mathbf{s}_{-,1}(\omega)+\sqrt{\gamma-4\kappa_g}\mathbf{s}_{-,2}^{(-)}(\omega)}{-i\omega+2\kappa_g}, \quad (44)$$

$$\mathbf{c}_{-,2}(\omega) = \frac{\sqrt{\gamma}\mathbf{s}_{-,2}(\omega)-\sqrt{\gamma-4\kappa_g}\mathbf{s}_{-,1}^{(-)}(\omega)}{-i\omega+2\kappa_g}, \quad (45)$$

$$\mathbf{c}_{+,1}(\omega) = \frac{\sqrt{\gamma}}{-i\omega+\gamma}\,\mathbf{s}_{+,1}(\omega), \quad (46)$$

$$\mathbf{c}_{+,2}(\omega) = -\frac{\sqrt{\gamma}}{i\omega}\left[\mathbf{s}_{+,2}(\omega)-\mathbf{s}_{+,1}^{(-)}(\omega)\right]. \quad (47)$$

Let us first analyze the fluctuations of the phases of the intracavity modes, which are the quantities analyzed in the main text. Being the output radiation in a vacuum state, we have

$$\langle\mathbf{s}_{\pm,i}(t)\mathbf{s}_{\pm,i}(t')\rangle=\frac{1}{4}\delta(t-t'),\quad i=1,2. \quad (48)$$

In addition, Eqs. (27)–(30) imply

$$\langle\mathbf{s}_{\pm,i}^{(-)}(t)\mathbf{s}_{\pm,i}^{(-)}(t')\rangle=\frac{1}{4}\delta(t-t'),\quad i=1,2, \quad (49)$$

so that Eq. (44) yields

$$\langle\mathbf{c}_{-,1}(\omega)\mathbf{c}_{-,1}^\dagger(\omega')\rangle=\frac{\gamma-4\kappa_g}{2\left(\omega^2+4\kappa_g^2\right)}2\pi\delta(\omega-\omega'). \quad (50)$$

The fluctuations of the difference of the phases of the emitted radiation are the difference between the fluctuations of the in-quadrature components of the intracavity mode amplitude divided by the average mode amplitude $a_0=b_0=\sqrt{P/(\gamma\hbar\omega_0)}$ where P is the average output power per mode, that is $\Delta\boldsymbol{\varphi}=$

$\sqrt{\gamma\hbar\omega/P}\,[\sqrt{2}\mathbf{c}_{-,1}(\omega)]$ so that the spectrum of the fluctuations of the phase difference is

$$\langle\Delta\boldsymbol{\varphi}(\omega)\Delta\boldsymbol{\varphi}^{\dagger}(\omega')\rangle=\frac{\hbar\omega_0\gamma(\gamma-4\kappa_g)}{P(\omega^2+4\kappa_g^2)}2\pi\delta(\omega-\omega'),\tag{51}$$

which is the result given in the main text [1], with a small correction arising from the fact that the analysis presented here accounted for the non-hermiticity of the mode coupling. Notice that the term $2\pi\delta(\omega-\omega')$ appearing here and in all other spectra can be removed by integration over frequency $f'=\omega'/(2\pi)$. This procedure returns for any given spectrum $\langle\mathbf{x}(\omega)\mathbf{x}(\omega')\rangle$ the Fourier transform of $\langle\mathbf{x}(t)\mathbf{x}(0)\rangle$, that is, if $\mathbf{x}(t)$ is a stationary process, the power spectrum of $\mathbf{x}(t)$.

The amplitude of the emitted radiation is given by

$$\mathbf{r}_a(t) = -\mathbf{s}_a(t)+\sqrt{\gamma}\,\mathbf{a}(t),\tag{52}$$

$$\mathbf{r}_b(t) = -\mathbf{s}_b(t)+\sqrt{\gamma}\,\mathbf{b}(t),\tag{53}$$

so that defining

$$\mathbf{r}_{c_\pm}(t)=\frac{1}{\sqrt{2}}\left[\mathbf{r}_a(t)\pm\mathbf{r}_b(t)\right],\tag{54}$$

we obtain for the fluctuations

$$\delta\mathbf{r}_{c_+}(t) = -\mathbf{s}_+(t)+\sqrt{\gamma}\,\mathbf{c}_+(t),\tag{55}$$

$$\delta\mathbf{r}_{c_-}(t) = -\mathbf{s}_-(t)+\sqrt{\gamma}\,\mathbf{c}_-(t).\tag{56}$$

The quadratures of the emitted radiation are readily obtained entering Eqs. (44)–(47) into Eqs. (55) and (56)

$$\delta\mathbf{r}_{c_-,1}(\omega) = \frac{\gamma\,\mathbf{s}_{-,1}(\omega)+\sqrt{\gamma(\gamma-4\kappa_g)}\,\mathbf{s}^{(-)}_{-,2}(\omega)}{-i\omega+2\kappa_g}-\mathbf{s}_{-,1}(\omega),\tag{57}$$

$$\delta\mathbf{r}_{c_-,2}(\omega) = \frac{\gamma\,\mathbf{s}_{-,2}(\omega)-\sqrt{\gamma(\gamma-4\kappa_g)}\,\mathbf{s}^{(-)}_{-,1}(\omega)}{-i\omega+2\kappa_g}-\mathbf{s}_{-,2}(\omega),\tag{58}$$

$$\delta\mathbf{r}_{c_+,1}(\omega) = \frac{i\omega}{-i\omega+\gamma}\,\mathbf{s}_{+,1}(\omega),\tag{59}$$

$$\delta\mathbf{r}_{c_+,2}(\omega) = -\frac{\gamma}{i\omega}\left[\mathbf{s}_{+,2}(\omega)-\mathbf{s}^{(-)}_{+,1}(\omega)\right]-\mathbf{s}_{+,2}(\omega).\tag{60}$$

As a consistency check, using that

$$[\mathbf{s}_{\pm,1}(\omega),\mathbf{s}^{\dagger}_{\pm,1}(\omega')]=[\mathbf{s}_{\pm,2}(\omega),\mathbf{s}^{\dagger}_{\pm,2}(\omega')]=0,\tag{61}$$

$$[\mathbf{s}_{\pm,1}(\omega),\mathbf{s}^{\dagger}_{\pm,2}(\omega')]=\frac{1}{4}\,[2\pi\delta(\omega-\omega')],\tag{62}$$

$$[\mathbf{s}^{(-)}_{\pm,1},\mathbf{s}^{(-)\dagger}_{\pm,1}(\omega')]=[\mathbf{s}^{(-)}_{\pm,2},\mathbf{s}^{(-)\dagger}_{\pm,2}(\omega')]=0,\tag{63}$$

$$[\mathbf{s}^{(-)}_{\pm,1},\mathbf{s}^{(-)\dagger}_{\pm,2}(\omega')]=\frac{1}{4}\,[2\pi\delta(\omega-\omega')],\tag{64}$$

one may show right away that

$$[\delta\mathbf{r}_{c_\pm,1}(\omega),\delta\mathbf{r}^{\dagger}_{c_\pm,1}(\omega)]=[\delta\mathbf{r}_{c_\pm,2}(\omega),\delta\mathbf{r}^{\dagger}_{c_\pm,2}(\omega)]=0,\tag{65}$$

$$[\delta\mathbf{r}_{c_\pm,1}(\omega),\delta\mathbf{r}^{\dagger}_{c_\pm,2}(\omega')]=\frac{1}{4}\,[2\pi\delta(\omega-\omega')],\tag{66}$$

so that the above equations correctly describe supermodes that are independent waves with bosonic commutation rules.

Using once again the correlation functions of the noise terms (61)–(64) we obtain

$$\langle\delta\mathbf{r}_{c_-,2}(\omega)\delta\mathbf{r}^{\dagger}_{c_-,2}(\omega')\rangle = \frac{1}{4}\left[\frac{2\gamma(\gamma-4\kappa_g)}{\omega^2+4\kappa_g^2}+1\right]2\pi\delta(\omega-\omega'),\tag{67}$$

$$\langle\delta\mathbf{r}_{c_+,2}(\omega)\delta\mathbf{r}^{\dagger}_{c_+,2}(\omega')\rangle = \frac{1}{4}\left(\frac{2\gamma^2}{\omega^2}+1\right)2\pi\delta(\omega-\omega').\tag{68}$$

The fluctuations of the difference of the phases of the emitted radiation $\Delta\boldsymbol{\varphi}_{\rm out}(\omega)$ is the difference of the fluctuations of the in-quadrature components divided by the amplitude of the output per mode in photon units $\sqrt{P/(\hbar\omega_0)}$, that is $\Delta\boldsymbol{\varphi}_{\rm out}=\sqrt{\hbar\omega_0/P}\,[\sqrt{2}\,\delta\mathbf{r}_{c_-,2}]$ so that the spectrum of the phase difference is $\langle\Delta\boldsymbol{\varphi}_{\rm out}(\omega)\Delta\boldsymbol{\varphi}^{\dagger}_{\rm out}(\omega')\rangle=2(\hbar\omega_0/P)\langle\delta\mathbf{r}_{c_-,2}(\omega)\delta\mathbf{r}^{\dagger}_{c_-,2}(\omega')\rangle$, that is

$$\langle\Delta\boldsymbol{\varphi}_{\rm out}(\omega)\Delta\boldsymbol{\varphi}^{\dagger}_{\rm out}(\omega')\rangle=\frac{\hbar\omega_0}{2P}\left[\frac{2\gamma(\gamma-4\kappa_g)}{\omega^2+4\kappa_g^2}+1\right]2\pi\delta(\omega-\omega').\tag{69}$$

The spectra of the phase fluctuations of the beat between the intracavity fields Eq. (51) and that of the output waves Eq. (69) differ primarily in the region $\omega\gg\gamma$, where the spectrum of the emitted radiation follows the phase fluctuations of the vacuum reflected from the cavity and the variance of the phase fluctuations of the beat are the sum of the variances of the phase fluctuations of two coherent states.

If we define $\boldsymbol{\varphi}_a=\delta\mathbf{r}_{a,2}/a_0$ and $\boldsymbol{\varphi}_b=\delta\mathbf{r}_{b,2}/b_0$ as the deviation of the phases of the emitted radiation from the steady state and use that $a_0=b_0$ we obtain $\langle\boldsymbol{\varphi}_a(\omega)\boldsymbol{\varphi}^{\dagger}_a(\omega')\rangle=\langle\boldsymbol{\varphi}_b(\omega)\boldsymbol{\varphi}^{\dagger}_b(\omega')\rangle$ with

$$\langle\boldsymbol{\varphi}_a(\omega)\boldsymbol{\varphi}^{\dagger}_a(\omega')\rangle = \frac{\hbar\omega_0}{2P}\left[\frac{\gamma^2}{\omega^2}+\frac{\gamma(\gamma-4\kappa_g)}{\omega^2+4\kappa_g^2}+1\right]2\pi\delta(\omega-\omega'),\tag{70}$$

$$\langle\boldsymbol{\varphi}_a(\omega)\boldsymbol{\varphi}^{\dagger}_b(\omega')\rangle = \frac{\hbar\omega_0}{2P}\left[\frac{\gamma^2}{\omega^2}-\frac{\gamma(\gamma-4\kappa_g)}{\omega^2+4\kappa_g^2}\right]2\pi\delta(\omega-\omega').\tag{71}$$

Three spectral regions are present. In the locking region $|\omega|\ll 2\kappa_g$, the phase fluctuations of the two modes are fully correlated with $\langle\boldsymbol{\varphi}_a(\omega)\boldsymbol{\varphi}^{\dagger}_a(\omega')\rangle\simeq\langle\boldsymbol{\varphi}_a(\omega)\boldsymbol{\varphi}^{\dagger}_b(\omega')\rangle$. In this spectral region, the variance of the phase fluctuations of each mode is one half of the free-running phase fluctuations of independent modes with the same output power and, similarly to the mode-locking case [2, 3], equal to the phase fluctuations of a single mode whose power is equal to the total power emitted by the laser. For $2\kappa_g<|\omega|<\gamma$, the two modes are unlocked and the phase fluctuations are the same of two free running modes of a laser which follow the Schawlow–Townes formula. For $|\omega|\gg\gamma$ the phase fluctuations are those of a radiation in a coherent state, as expected because they are the shot-noise fluctuations of the vacuum field reflected by the laser cavity outside its frequency cutoff. The expressions of the frequency noise spectra of the mode beat and of the two counterpropagating mode can be readily obtained multiplying by ω^2 the corresponding phase noise spectra.

Let us now analyze the amplitude fluctuations. We have

$$\langle\delta\mathbf{r}_{c_+,1}(\omega)\delta\mathbf{r}^{\dagger}_{c_+,1}(\omega')\rangle = \frac{\omega^2}{4\,[\omega^2+\gamma^2]}2\pi\delta(\omega-\omega'),\tag{72}$$

$$\langle\delta\mathbf{r}_{c_-,1}(\omega)\delta\mathbf{r}^{\dagger}_{c_-,1}(\omega')\rangle = \frac{1}{4}\left[\frac{2\gamma(\gamma-4\kappa_g)}{\omega^2+4\kappa_g^2}+1\right]2\pi\delta(\omega-\omega'),\tag{73}$$

and consequently $\langle \delta \mathbf{r}_{a,1}(\omega)\delta \mathbf{r}^{\dagger}_{a,1}(\omega')\rangle = \langle \delta \mathbf{r}_{b,1}(\omega)\delta \mathbf{r}^{\dagger}_{b,1}(\omega')\rangle$ and

$$\langle \delta \mathbf{r}_{a,1}(\omega)\delta \mathbf{r}^{\dagger}_{a,1}(\omega')\rangle = \frac{1}{8}\left[\frac{2\gamma(\gamma-4\kappa_g)}{\omega^2+4\kappa_g^2} - \frac{\gamma^2}{\omega^2+\gamma^2} + 2\right] 2\pi\delta(\omega-\omega'), \tag{74}$$

$$\langle \delta \mathbf{r}_{a,1}(\omega)\delta \mathbf{r}^{\dagger}_{b,1}(\omega')\rangle = -\frac{1}{8}\left[\frac{2\gamma(\gamma-4\kappa_g)}{\omega^2+4\kappa_g^2} + \frac{\gamma^2}{\omega^2+\gamma^2}\right] 2\pi\delta(\omega-\omega'). \tag{75}$$

For $|\omega| \ll \gamma$, similarly to the amplitude squeezing of the radiation emitted from the laser when pump fluctuations are suppressed [4, 5], the fluctuations of the sum of the amplitudes of the two modes (the fluctuations of the amplitude of the supermode) are below the quantum noise limit (sub-Poissonian) and zero at $\omega = 0$. The amplitudes of the two modes are locked, with a finite variance, for $\omega \ll 2\kappa_g$, and their fluctuations are correlated. For $2\kappa_g < |\omega| < \gamma$, the two modes are unlocked and their amplitudes experience partition noise, while the fluctuations of the sum of their amplitude are still suppressed. For $|\omega| \gg \gamma$, above the cutoff introduced by the laser cavity, the amplitude fluctuations are those of a radiation in a coherent state, because they are those of the vacuum state reflected from the cavity.

It is interesting to discuss the autocorrelation function of the phase fluctuations of the beat of the output fields. Let us suppose that the measurement is performed with a finite bandwidth B, by assuming an ideal square low-pass filter of bandwidth B with a flat unit response for $|\omega|/(2\pi) \le B/2$ and zero outside. This situation describes, for instance, an ideal measurement with a sampling period $T_{\text{sampling}} = 1/B$. Then, integration over ω' in the two-dimensional inverse Fourier transform of Eq. (69) produces a result that depends only on $T = t' - t$. For $B \gg 2\kappa_g$, we may neglect the effect of frequency filtering on the first term by approximating, in the convolution with this term, the sinc generated by the spectral filtering with a Dirac delta function. After doing so, another inverse Fourier transformation with respect to ω produces

$$\langle \Delta\boldsymbol{\varphi}_{\text{out}}(t+T)\Delta\boldsymbol{\varphi}_{\text{out}}(t)\rangle = \frac{\hbar\omega_0}{2P}\left[\frac{\gamma(\gamma-4\kappa_g)}{2\kappa_g}\exp\left(-2\kappa_g T\right) + B\,\frac{\sin\left(\pi B T\right)}{\pi B T}\right]. \tag{76}$$

The sinc appearing in this expression represents the effect of the filtered vacuum noise reflected from the laser cavity or, in a semiclassical language, the shot noise of the detection. Using now Eq. (76) in the expression for the Allan variance in terms of the time autocorrelation function

$$\sigma_T^2 = \frac{1}{T^2}\left[3\left\langle \Delta\boldsymbol{\varphi}_{\text{out}}(t)^2\right\rangle - 4\left\langle \Delta\boldsymbol{\varphi}_{\text{out}}(t+T)\Delta\boldsymbol{\varphi}_{\text{out}}(t)\right\rangle + \left\langle \Delta\boldsymbol{\varphi}_{\text{out}}(t+2T)\Delta\boldsymbol{\varphi}_{\text{out}}(t)\right\rangle\right], \tag{77}$$

and assuming that T is a multiple of the sampling period if the filtering is the effect of sampling, or in general that $T \gg 1/B$, we obtain

$$\sigma_T^2 = \frac{\hbar\omega_0}{2PT^2}\left[\frac{\gamma(\gamma-4\kappa_g)}{2\kappa_g}\left[3 - 4\exp\left(-2\kappa_g T\right) + \exp\left(-4\kappa_g T\right)\right] + 3B\right]. \tag{78}$$

For $\kappa_g T \gg 1$ we have

$$\sigma_T^2 = \frac{3\hbar\omega_0}{2PT^2}\left[\frac{\gamma(\gamma-4\kappa_g)}{2\kappa_g} + B\right]. \tag{79}$$

The term proportional to B is the effect that extends to long T of the high frequency portion of the vacuum noise fluctuations reflected by the laser and coherently added to the emitted light beams. This contribution is negligible for $B \ll \gamma(\gamma-4\kappa_g)/(2\kappa_g)$. Equations (78) and (79) are the same expressions given in the main text, with the addition of the shot noise contribution and with a small correction arising from the interference between the emitted radiation and the vacuum field reflected from the cavity.

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